

On quantum group symmetry and Bethe ansatz for the asymmetric twin spin chain with integrable boundary

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Abstract

Motivated by a study of the crossing symmetry of the asymmetric twin or ‘gemini’ representation of the affine Hecke algebra we give a construction for *crossing tensor space representations* of ordinary Hecke algebras. These representations build solutions to the Yang–Baxter equation satisfying the crossing condition (that is, integrable quantum spin chains). We show that *every* crossing representation of the Temperley–Lieb algebra appears in this construction, and in particular that this construction builds *new* representations. We extend these to new representations of the blob algebra, which build new solutions to the Boundary Yang–Baxter equation (i.e. open spin chains with integrable boundary conditions).

We prove that the open spin chain Hamiltonian derived from Sklyanin’s commuting transfer matrix using such a solution can always be expressed as the representation of an element of the blob algebra, and determine this element. We determine the representation theory (irreducible content) of the new representations and hence show that all such Hamiltonians have the same spectrum up to multiplicity, for any given value of the algebraic boundary parameter. (A corollary is that our models have the same spectrum as the open XXZ chain with nondiagonal boundary — despite differing from this model in having reference states.) Using this multiplicity data, and other ideas, we investigate the underlying quantum group symmetry of the new Hamiltonians. We derive the form of the spectrum and the Bethe ansatz equations.

LAPTH-1089/05

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1 Introduction

The integrability of quantum field and lattice theories in two dimensions is closely tied to the factorization of multi-particle scattering [1–4]. Let $R_{ij}(\lambda)$ describe the scattering amplitude for particles i, j with incidence angle λ . The Yang–Baxter equation [1, 5–10]

$$R_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) R_{13}(\lambda_1) R_{12}(\lambda_1 - \lambda_2) \quad (1)$$

provides a key set of constraints on possible forms of $R_{ij}(\lambda)$ consistent with factorization. When non-trivial boundaries are present (e.g. in field theories on a half line), $R_{ij}(\lambda)$ must also satisfy the boundary Yang–Baxter (reflection) equation [11, 12]:

$$R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) = K_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2). \quad (2)$$

Here the K -matrix is the boundary scattering matrix of the theory.

The physical importance of integrable systems with boundary has driven sustained interest in the study of solutions to these equations (some key references are [13–25]). In [26] the structural similarity between these equations and the cylinder braid group relations is exploited to derive solutions systematically. In particular, we considered the quotient of the cylinder braid group algebra called the blob algebra, here denoted $b_n(q, m)$. This is a two-parameter extension to the Temperley-Lieb algebra $T_n(q)$. The Temperley-Lieb algebra provides a universal approach to the Yang–Baxter equation in the sense that any tensor space representation $\rho : T_n(q) \rightarrow \text{End}(V^{\otimes n})$ gives a solution to (1) via a standard construction (see later), and furthermore the Hamiltonian spectrum of each model constructed in this way is the same (up to multiplicities which may be computed in representation theory). Suppose ρ extends to a representation of b_n . Then an analogous construction provides a K -matrix solving the reflection equation (2) [26, 27], and there is an analogous equivalence among corresponding models.

The simple extension of the ordinary tensor space representation of T_n to b_n which is described in [28, §4] gives the open spin-1/2 XXZ chain with nondiagonal boundary conditions (see also [16]). This is perhaps the obvious model to start with when studying boundary conditions, but it has some significant limitations (see later). In [26] we introduced a model based on an asymmetrically cabled spin-chain-like representation Θ of T_n , and a particular extension of Θ to b_n . The representation is well defined, and hence the model is integrable, for all values of the parameters q and m (m is a boundary parameter). Although the model describes a system of interacting spins in the same way as the XXZ model [2, 3], it provides a framework for treating the effect of boundaries which is significantly different from previous approaches. In particular some of the properties of XXZ often invoked in implementing the Bethe ansatz (such as symmetry properties) do not hold in the usual way. Then again, this model has simple reference states, while the usual open chain with nondiagonal boundary does not (although important progress has been made towards determining the spectrum of the ordinary open chain recently [29] despite the lack of obvious reference states). We prove here that the ordinary open chain and our new model *with* reference states are equivalent. This will allow us to take a line of lower resistance through Bethe ansatz calculations than either model offers alone.

The new model turns out to offer a more general treatment of boundaries than previously possible, so this motivates us to seek equivalents in this setting for the symmetry properties of XXZ. The first approach to this problem was through algebraic Lie theory [30], but there only abstract (although intriguing) results about the symmetry of the original extension were found. Here we look directly at the Bethe ansatz, and the most general possible extension.

Our objectives in *this* paper are firstly to put Θ in a more general setting by considering ‘crossing tensor space representations’ (which ensure a crossing condition necessary for Bethe ansatz); then to describe extensions of Θ from T_n to b_n systematically; then for each of the types of representation of b_n which we find: (i) to examine the symmetry algebra of the Hamiltonian of the resultant spin-chain; (ii) to investigate the Bethe Ansatz of these Hamiltonians. The main results in this regard are the symmetries summarized in Propositions 8, 9 and equations (64), (89), (95), and the form of the Bethe Ansatz solution in (127).

In section 2 we discuss the role and implementation of crossing symmetry in the algebraic construction. In section 3 we recall the definition of Θ , derive the new representations, and establish some important notation (swept under the carpet in [26]). We then derive the universal algebraic Hamiltonian. In section 4 we recall the role of quantum groups in the symmetry of ordinary spin chains, and discuss how this might generalize to our case. We find a number of actions of quantum groups on our chain, but not the complete symmetry algebra. We discuss how one might proceed to find the complete symmetry, and illustrate some subtleties compared to the XXZ spin chain case [2, 3]. The remainder of the paper is concerned with the solution of the Bethe Ansatz for the most physically interesting cases.

A comment is in order on the physical motivation for this approach. The aim is to understand and compute with directly physically relevant models [31]. However, not every directly physically relevant model is integrable, and those integrable models, such as XXZ, which do have arguable physical relevance do not in general remain integrable for arbitrary boundary conditions (or, if they do, present significant technical problems). The idea here is to consider models which *are* integrable with suitably general boundary conditions, sacrificing the direct superficial similarity with XXZ. However we then *prove* these models to have the same spectrum (up to multiplicity) as more manifestly physically relevant models.

1.1 The blob algebra and the boundary YBE

Definition 1 *Let q, δ_e, κ be given scalars, and $\delta = -q - q^{-1}$. The blob algebra $b_n = b_n(q, \delta_e, \kappa)$ is defined by generators $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_{n-1}$ and \mathbf{e} , and relations [28]:*

$$\mathcal{U}_i \mathcal{U}_i = \delta \mathcal{U}_i \tag{3}$$

$$\begin{aligned} \mathcal{U}_i \mathcal{U}_{i+1} \mathcal{U}_i &= \mathcal{U}_i \\ [\mathcal{U}_i, \mathcal{U}_j] &= 0, \quad |i - j| \neq 1 \end{aligned} \tag{4}$$

(so far we have the ordinary Temperley–Lieb algebra $T_n(q)$ [32])

$$\mathbf{e} \mathbf{e} = \delta_e \mathbf{e} \tag{5}$$

$$\mathcal{U}_i \mathbf{e} \mathcal{U}_i = \kappa \mathcal{U}_i \quad (6)$$

$$\begin{bmatrix} \mathcal{U}_i, \mathbf{e} \end{bmatrix} = 0, \quad i \neq 1. \quad (7)$$

Note that we are free to renormalize \mathbf{e} , changing only δ_e and κ (by the same factor), thus from δ, δ_e, κ there are really only two relevant parameters (originally [26] this freedom was used to fix $\delta_e = 1$, but this is not generally the best choice). It will be natural later on to reparameterize so that the three are related (they only depend on q and a single additional parameter m), but it is convenient to treat them separately for the moment, and leave m hidden.

Let $\rho : T_n(q) \rightarrow \text{End}(V^{\otimes n})$ be a tensor space representation of $T_n(q)$ (see section 2), \mathcal{P} be the permutation operator on $V \otimes V$, and $q = e^{i\mu}$. Then a solution to (1) is given by [8]

$$R_{i \ i+1}(\lambda) = \mathcal{P}_{i \ i+1}(\sinh \mu(\lambda + i) \rho(1) + \sinh \mu \lambda \rho(\mathcal{U}_i)). \quad (8)$$

Suppose ρ extends to a representation of b_n . Then the analogous construction for a K -matrix solving the reflection equation (2) is [26, 27]

$$K(\lambda) = x(\lambda) \rho(1) + y(\lambda) \rho(\mathbf{e}) \quad (9)$$

where

$$x(\lambda) = -\delta_e \cosh \mu(2\lambda + i) - \kappa \cosh 2\mu\lambda - \cosh 2i\mu\zeta \quad y(\lambda) = 2 \sinh 2\mu\lambda \sinh i\mu \quad (10)$$

(here ζ is an arbitrary constant).

2 The Temperley-Lieb algebra and tensor space

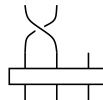
Although we will be concerned with using the blob algebra to treat boundary solutions, it is helpful to begin by unpacking a little what is meant by a *tensor space* representation, and the crossing symmetry condition.

2.1 Preliminaries

The next few results are elementary. Let k be a field, N a natural number and V an N -dimensional k -vector space with basis $B = \{b_1, b_2, \dots, b_N\}$. The permutation matrix \mathcal{P}_{ij} acts on $V^{\otimes n}$ by

$$\mathcal{P}_{ij} v_1 \otimes \dots \otimes v_i \otimes \dots \otimes v_j \otimes \dots = v_1 \otimes \dots \otimes v_j \otimes \dots \otimes v_i \otimes \dots \quad (v_k \in V)$$

A standard picture of this is that given a *chip* with rows of n legs, the action of \mathcal{P}_{ij} is to cross the corresponding legs:

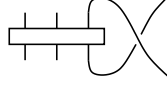


(here the over/under information is unimportant).

For $M \in \text{End}(V^{\otimes n})$ the i^{th} factor transpose is defined by

$$\langle \dots v'_i \dots | M^{t_i} | \dots v_i \dots \rangle = \langle \dots v_i \dots | M | \dots v'_i \dots \rangle \quad (v_k \in B)$$

(so the usual total transpose is $t = t_1 t_2 \dots t_n$). In the chip realisation, t_i is a kind of *s-channel crossing*:



Proposition 1 For any i, j

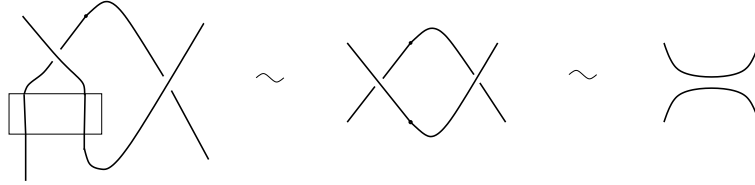
$$\mathcal{P}_{ij}^{t_i} = \mathcal{P}_{ij}^{t_j}$$

$$\langle v'_1 v'_2 | \mathcal{P}_{ij}^{t_j} | v_1 v_2 \rangle = \delta_{v_1 v_2} \delta_{v'_1 v'_2}$$

The proof is elementary (if slightly tedious). An example is more enlightening:

$$\begin{array}{ccc} \left(\begin{array}{c|cc} 1 & & \\ \hline & 0 & 1 \\ \hline & 1 & 0 \\ & & 1 \end{array} \right) & \xrightarrow{t_2} & \left(\begin{array}{ccc} 1 & & 1 \\ & 0 & 0 \\ & 0 & 0 \\ 1 & & 1 \end{array} \right) \\ & \searrow t_1 & \\ & \left(\begin{array}{ccc} 1 & & 1 \\ & 0 & 0 \\ & 0 & 0 \\ 1 & & 1 \end{array} \right) & \end{array}$$

The composite index contraction $\mathcal{P}_{ij}^{t_j}$ may be represented pictorially by



(NB, this index contraction should not be confused with a Temperley-Lieb diagram — see later). A trivial corollary is

$$\mathcal{P}_{ij} \mathcal{P}_{ij}^{t_j} = \mathcal{P}_{ij}^{t_j} \quad (11)$$

Proposition 2 For any $\mathcal{V} \in \text{End}(V)$

$$(\mathcal{V} \otimes \mathbb{I}) \mathcal{P}_{12}^{t_1} = (\mathbb{I} \otimes \mathcal{V})^t \mathcal{P}_{12}^{t_1}$$

The proof is a direct calculation. The picture for this is



(i.e. \mathcal{V} is a black dot and \mathcal{V}^t a white dot).

2.2 Realisations of $T_n(q)$

Theorem 1 *Provided that*

$$\mathcal{V}^t \mathcal{V}^t \mathcal{V} \mathcal{V} = \mathbb{I} \quad (12)$$

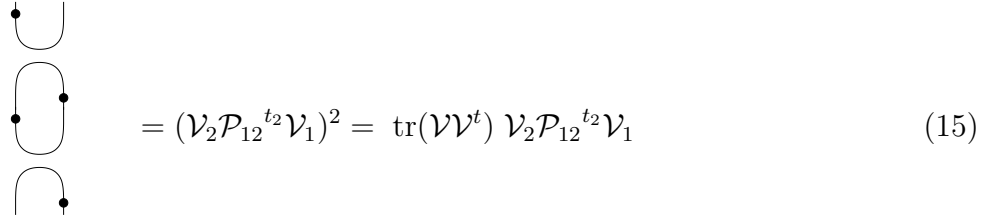
$$\text{tr}(\mathcal{V} \mathcal{V}^t) = -(q + q^{-1}) \quad (13)$$

then

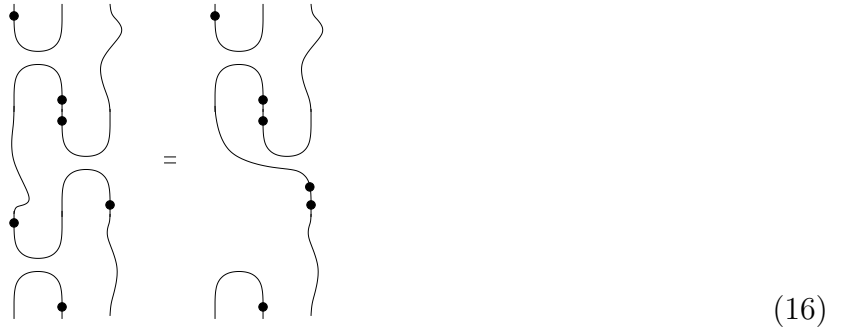
$$\rho_{\mathcal{V}}(\mathcal{U}_i) = \mathcal{V}_{i+1} \mathcal{P}_{i+1}{}^{t_{i+1}} \mathcal{V}_i \quad (14)$$

is a representation of $T_n(q)$ on $V^{\otimes n}$ for any n .

Proof: Consider the following identities:



$$= (\mathcal{V}_2 \mathcal{P}_{12}^{t_2} \mathcal{V}_1)^2 = \text{tr}(\mathcal{V} \mathcal{V}^t) \mathcal{V}_2 \mathcal{P}_{12}^{t_2} \mathcal{V}_1 \quad (15)$$



$$= \quad (16)$$

□

Remark: These pictures are a departure from the standard diagram calculus for the Temperley-Lieb algebra. In the calculus the lines represent more than a simple index contraction, and decorations are superfluous. Here the lines *are* merely index contractions (so, without the decorations we have only a realisation of $T_n(q)$ with $q + q^{-1} = N$, i.e. *classical* Temperley-Lieb in case $N = 2$).¹

Definition 2 *A representation ρ of $T_n(q)$ is a local tensor space representation if it acts on the tensor product $V^{\otimes n}$ for some V and*

$$\rho(\mathcal{U}_{i+1}) = \mathcal{P}_{i+1} \mathcal{P}_{i+1}{}^{t_{i+1}} \rho(\mathcal{U}_i) \mathcal{P}_{i+1}{}^{t_{i+1}} \mathcal{P}_{i+1}$$

and $\rho(\mathcal{U}_i)$ acts trivially on all but the i^{th} and $i + 1$ st tensor factor.

¹These remarks also serve to differentiate between the construction here and similar looking diagrams for 'wreath' Temperley-Lieb algebras such as in [33]. None-the-less, this decorated form is familiar in some long standing diagrammatic treatments of the Temperley-Lieb algebra, and it has been tacitly assumed that choices for \mathcal{V} correspond to variations on the XXZ representation (which commutes with the action of $U_q(\mathfrak{sl}(2))$ on tensor space when $N = 2$) by a combination of cabling (i.e. $U_q(\mathfrak{sl}(2))$ related) and similarity transformations. It turns out that the construction is more general! (It has not widely been considered likely that there *are* tensor space representations outside the $U_q(\mathfrak{sl}(2))$ construction. So we should confess that we would have overlooked the fact again, were it not for some serendipity in our investigation of the crossing properties of the representation introduced in [34].)

A tensor space R -matrix $R_{ij} = \mathcal{P}_{i+1\ j} R_{i\ i+1} \mathcal{P}_{i+1\ j}$ is said to satisfy the *unitarity* condition if

$$R_{ij}(\lambda)R_{ji}(-\lambda) \propto \mathbb{I} \quad (17)$$

and said to satisfy the *crossing* condition [12, 35] if

$$\mathcal{V}_i R_{ij}^{t_j}(\lambda) \mathcal{V}_i = -R_{ij}(-\lambda - i) \quad (18)$$

for some matrix $\mathcal{V} \in \text{End}(V)$.

Theorem 2 *Let $R_{ij} = \mathcal{P}_{i+1\ j} R_{i\ i+1} \mathcal{P}_{i+1\ j}$ be constructed as in (8), using the \mathcal{V} -construction (14):*

$$R_{ij}(\lambda) = \sinh(\mu(\lambda + i)) \mathcal{P}_{ij} + \sinh(\mu\lambda) \mathcal{P}_{ij} \mathcal{V}_j \mathcal{P}_{ij}^{t_j} \mathcal{V}_i \quad (19)$$

*with $\mathcal{V}_i^2 = \mathbb{I}$. Then R_{ij} solves the Yang-Baxter equations; obeys the unitarity condition; and has the crossing condition property.*²

(For this reason a representation of form $\rho_{\mathcal{V}}$ is called a crossing representation.)

Conversely, suppose that ρ is a tensor space representation of $T_n(q)$ on $V^{\otimes n}$ (any V), and that the crossing property holds with crossing matrix \mathcal{V} at $\lambda = 0$:

$$\mathcal{V}_i R_{i\ i+1}^{t_{i+1}}(0) \mathcal{V}_i = -R_{i\ i+1}(-i) \quad (20)$$

Then

$$\rho(\mathcal{U}_i) = \mathcal{P}_{i\ i+1} \mathcal{V}_i \mathcal{P}_{i\ i+1}^{t_{i+1}} \mathcal{V}_i = \mathcal{V}_{i+1} \mathcal{P}_{i\ i+1}^{t_{i+1}} \mathcal{V}_i$$

In other words ρ can always be considered to be determined by \mathcal{V} .

Proof: The TL relations already imply a solution to YBE — this is well known (see §1.1). The solution to the unitarity condition is a direct consequence of the TL relation (3) in particular. Now, however, if we use the \mathcal{V} -construction, since R_{ij} is given by (19) we have

$$\begin{aligned} \mathcal{V}_i R_{ij}^{t_j}(\lambda) \mathcal{V}_i &= \sinh(\mu(\lambda + i)) \mathcal{V}_i \mathcal{P}_{ij}^{t_j} \mathcal{V}_i + \sinh(\mu\lambda) \mathcal{V}_i (\mathcal{P}_{ij} \mathcal{V}_j \mathcal{P}_{ij}^{t_j} \mathcal{V}_i)^{t_j} \mathcal{V}_i \\ &\stackrel{(11)}{=} \sinh(\mu(\lambda + i)) \mathcal{P}_{ij} \mathcal{V}_j \mathcal{P}_{ij}^{t_j} \mathcal{V}_i + \sinh(\mu\lambda) \mathcal{V}_i^2 \mathcal{P}_{ij} \mathcal{V}_i^2 \end{aligned}$$

Now apply $\mathcal{V}_i^2 = \mathbb{I}$ to get (18). The converse is a routine manipulation after substitution of (19) into (20). \square

2.3 Examples

A classification of solutions to the conditions $\mathcal{V}^2 = \mathbb{I}$ and (13) for general N is not our objective here (although it is an interesting problem). We will restrict ourselves to the examples we need. We are interested in solutions which are valid for a fixed but arbitrary value of q (or equivalently with q regarded as an indeterminate).

²Note that $\mathcal{V}_i^2 = \mathbb{I}$ implies (12). Thus sufficient conditions for integrability with parameter q are $\mathcal{V}_i^2 = \mathbb{I}$ and the trace condition.

Let u be such that $u^2 = -q$. The $\rho_{\mathcal{V}}$ construction is clearly trivial unless $N \geq 2$. In case $N = 2$, a solution to the conditions is given by

$$\mathcal{V} = \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix} \longrightarrow \mathcal{V}_2 \mathcal{P}_{12}{}^{t_2} \mathcal{V}_1 = \begin{pmatrix} 0 & & & \\ & -q & 1 & \\ & 1 & -q^{-1} & \\ & & & 0 \end{pmatrix} =: U(q) \quad (21)$$

That is, with $V = \mathbb{C}^2$,

$$\mathcal{R}_q(\mathcal{U}_l) := \rho_{\mathcal{V}}(\mathcal{U}_l) = 1 \otimes 1 \otimes \dots \otimes U(q) \otimes \dots \otimes 1 \otimes 1 \in \text{End}(V^n) \quad (22)$$

(acting non-trivially on $V_l \otimes V_{l+1}$) defines a representation of $T_n(q)$ (any q, n). This is the representation arising in the n -site XXZ spin chain [8].

More generally, with

$$\mathcal{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the square condition gives $a^2 = d^2$, $b(a+d) = c(a+d) = 0$. In case $a = d > 0$ then $b = c = 0$ so there is no solution to the trace condition for q indeterminate. In case $a = -d$ then there is a solution

$$\mathcal{V} = \begin{pmatrix} \sqrt{-\alpha(\alpha + u + u^{-1})} & \alpha + u \\ \alpha + u^{-1} & -\sqrt{-\alpha(\alpha + u + u^{-1})} \end{pmatrix} \quad (23)$$

for each α . Note that the solution in (21) is the $\alpha = 0$ case. Since this space of solutions is continuous with (21) it follows that the representations of $T_n(q)$ constructed are generically equivalent. (To see this note that characters will be continuous functions of α , but irreducible multiplicities are integers. The only continuous integer valued functions are constants, thus the irreducible content of the representation does not depend on α .)

We have shown that there is only one q -indeterminate class of solutions for $N = 2$, and that the representations of $T_n(q)$ arising are all generically equivalent.

It is worth remarking that (23) is the first example of a mixed state crossing matrix to be observed. By virtue of our previous remark this generalisation does not seem to be of particular intrinsic interest. The possibilities for higher N , however, are intriguing (but not considered further here).

The example which concerns us (because of its role in building representations of b_n) has $N = 4$. Noting the example above we restrict attention to antidiagonal \mathcal{V} . Then we have that $\mathcal{V} = \text{antidiagonal}(a, b, b^{-1}, a^{-1})$ and that $a^2 + a^{-2} + b^2 + b^{-2} = -(q + q^{-1})$. There are a number of distinct classes of solutions. In order to understand the significance of the so called *geminis* solution (given explicitly shortly) it is appropriate to recall its *original* construction [34], in which the crossing property appears to be just a happy accident.

3 The gemini representations and spin chains

Define

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{e^{i\mu x} - e^{-i\mu x}}{e^{i\mu} - e^{-i\mu}} = \frac{\sinh(i\mu x)}{\sinh(i\mu)} \quad (24)$$

Set

$$r = i\sqrt{iq}, \quad \hat{r} = \sqrt{iq} \quad (25)$$

so that $r\hat{r} = -q$. The map $\Theta : T_n(q) \rightarrow \text{End}(V^{2n})$ is constructed by combining parts of the representations \mathcal{R}_r of $T_{2n}(r)$ and $\mathcal{R}_{\hat{r}}$ of $T_{2n}(\hat{r})$ as follows:

$$\Theta(\mathcal{U}_l) = \mathcal{R}_r(\mathcal{U}_{n-l}) \mathcal{R}_{\hat{r}}(\mathcal{U}_{n+l}) \quad (26)$$

It is easy to check that this is a representation. Because of the way it melds two chains with *different* quantum parameters this is called the asymmetric twin or gemini representation.

Note that the Temperley–Lieb algebra is *not* a bialgebra, so the realization (26) is not an obvious result. The meld gives a new kind of tensor space representation. It has been studied from the point of view of quasihereditary algebras in [30], but given the importance of the orthodox XXZ representation in integrable physics, and the properties of Θ we are about to elucidate, this representation merits more concrete study. (It is easy to see that this gemini idea is amenable to further massive generalization, but we will not pursue the point here.)

A consequence of the construction is that Θ is a crossing representation (the crossing matrix is given in (40)). Another striking feature of this representation is that it extends to an interesting representation of b_n in a number of distinct ways. It will be evident that $\Theta(\mathcal{U}_1)$ acts on $V^{\otimes 4} = V_{n-1} \otimes V_n \otimes V_{n+1} \otimes V_{n+2}$ as a rank 1 matrix. Consider for a moment the case $n = 2$:

$$\Theta(\mathcal{U}_1) = \frac{1}{-q} ((0, -r, 1, 0) \otimes (0, -\hat{r}, 1, 0))^t \cdot ((0, -r, 1, 0) \otimes (0, -\hat{r}, 1, 0)) \quad (27)$$

In consequence *any* matrix M acting non-trivially only on this $V^{\otimes 4}$ obeys $\Theta(\mathcal{U}_1)M\Theta(\mathcal{U}_1) = k^M\Theta(\mathcal{U}_1)$ for some k^M — cf. relation (6). On the other hand relation (7) is satisfied by any matrix $\Theta(\mathbf{e})$ acting non-trivially only on the middle two factors ($V_n \otimes V_{n+1}$) of this $V^{\otimes 4}$. We will call this the *local condition*. (NB, this is not a necessary condition, but our choice here — a consequence of this choice is that the solutions we consider will be characterised by a 4×4 matrix \mathcal{M} giving the action on $V_n \otimes V_{n+1}$). Finally (5) requires that the spectrum of any candidate for $\Theta(\mathbf{e})$ be taken from $\{\delta_e, 0\}$. We do not pursue the most general form for the extension here, but concentrate on a representative set of interesting choices (i.e. with characterising matrix \mathcal{M} of rank 1 or 2).

We have the following justification for this level of generality:

Proposition 3 *The spectrum of the spin chain model built from a representation of b_n of the form of Θ depends on \mathcal{M} only through the rank of \mathcal{M} .*

We will prove this result in section 4.5.

Fixing q , define matrices in $\text{End}(V^{\otimes 2n})$ as follows:

$$M^i(Q) = \frac{-\delta_e}{Q + Q^{-1}} 1 \otimes 1 \otimes \dots \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -Q & 1 & 0 \\ 0 & 1 & -Q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \dots \otimes 1 \otimes 1 \quad (28)$$

where the 4×4 matrix acts on $V_n \otimes V_{n+1}$ and Q is some scalar;

$$M^{ii}(Q) = \frac{-\delta_e}{Q + Q^{-1}} 1 \otimes 1 \otimes \dots \otimes \begin{pmatrix} -Q & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -Q^{-1} \end{pmatrix} \otimes \dots \otimes 1 \otimes 1 \quad (29)$$

$$M^+(Q, Q') = M^i(Q) + M^{ii}(Q') \quad (30)$$

$$M^{iii}(Q_1, Q_2) = \frac{\delta_e}{(Q_1 + Q_1^{-1})(Q_2 + Q_2^{-1})} 1 \otimes 1 \otimes \dots \otimes \begin{pmatrix} -Q_1 & 1 \\ 1 & -Q_1^{-1} \end{pmatrix} \otimes \begin{pmatrix} -Q_2^{-1} & 1 \\ 1 & -Q_2 \end{pmatrix} \otimes \dots \otimes 1 \otimes 1 \quad (31)$$

where the 2×2 matrices act separately on V_n, V_{n+1} respectively; and $M^{iii}(Q) = M^{iii}(i\sqrt{iQ}, \sqrt{iQ})$.

For $I \in \{i, ii, +, iii\}$ direct calculation shows

$$\Theta(\mathcal{U}_1) M^I \Theta(\mathcal{U}_1) = \kappa^I \Theta(\mathcal{U}_1)$$

where

$$\kappa^i(Q) = \frac{-\delta_e(q^{-1}Q + qQ^{-1})}{Q + Q^{-1}}, \quad \kappa^{ii}(Q) = \frac{-\delta_e(iQ - iQ^{-1})}{Q + Q^{-1}}, \quad (32)$$

$$\kappa^+(Q, Q') = \kappa^i(Q) + \kappa^{ii}(Q'), \quad \kappa^{iii}(Q) = \frac{-\delta_e(q^{-1}Q + qQ^{-1} + 2)}{Q + Q^{-1}} \quad (33)$$

so

Proposition 4 *For each $I \in \{i, ii, +, iii\}$ there is a representation $\Theta^I : b_n(q, \delta_e, \kappa) \rightarrow \text{End}(V^{\otimes 2n})$ given by $\Theta^I(\mathcal{U}_i) = \Theta(\mathcal{U}_i)$, $\Theta^I(\mathbf{e}) = M^I$, provided $\kappa^I(Q) = \kappa$. (NB, for given I this is a condition on Q .³)*

The representation Θ^i is the representation ρ_0 in [34]; while Θ^+ with $Q' \rightarrow 0$ is the representation ρ_x there. The other representations are new.

We will need to have to hand one more kind of representation of b_n . It will be evident that if $\kappa = 0$ then the quotient relation $\mathbf{e} = 0$ is consistent with the defining relations of b_n . It follows that for this particular value of κ any representation of T_n may be extended to a representation of b_n by representing \mathbf{e} by zero. In case we start with Θ , we will write Θ^0 for this extension to b_n . It will be evident from (9) that this gives a trivial K -matrix.

3.1 Anatomy of a spin chain

It will be convenient to be able to refer to individual factors in the tensor product $V^{\otimes N}$. Normally we write $V^{\otimes N} = V_1 \otimes V_2 \otimes \dots \otimes V_N$ so V_i is the space of the i^{th} spin in the spin chain. In our situation, however, with $N = 2n$, it will be convenient to relabel

$$V^{\otimes 2n} = V_{n-} \otimes V_{(n-1)-} \otimes \dots \otimes V_{1-} \otimes V_{1+} \otimes \dots \otimes V_{(n-1)+} \otimes V_{n+}.$$

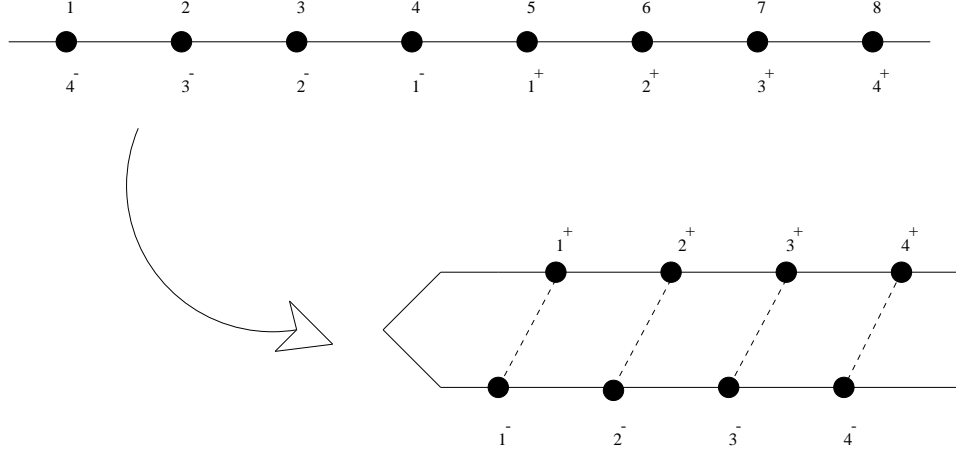


Figure 1: Conventional spin chain site numbering, and mirror image pair site numbering. (The new composite sites are shown clustered by dotted lines.) NB, Here the chain links merely induce an order on the vertices, convenient for labelling — they do not correspond to tensor index contractions.

I.e.

$$n + l \rightarrow l^+, \quad n - l + 1 \rightarrow l^-, \quad l = 1, \dots, n. \quad (34)$$

One may think of this as a product of a mirror image pair of factors $V^{\otimes n}$, as the new labelling suggests. One should then think of ‘folding up’ the linear chain at the point between V_{1-} and V_{1+} , so that it becomes a double thickness chain with composite sites of form $V_{i-} \otimes V_{i+}$. Thus in particular the composite site $V_{1-} \otimes V_{1+}$ lies at one *end* of the chain, and \mathbf{e} acts non-trivially only on this boundary site. All of these points are illustrated by figure 1.

This labelling issue is mundane, but it is practically significant here. Let $\mathbf{V} = (V^{\otimes N}, (l_1, l_2, \dots, l_N))$ be a tensor power of a vector space V together with a labelling scheme for the tensor factors. Then for each M acting on $V^{\otimes m}$ ($m \leq N$) and list $\{i_1, i_2, \dots, i_m\} \subseteq \{l_1, l_2, \dots, l_N\}$ we define $\mathcal{I}_{i_1, i_2, \dots, i_m}^{\mathbf{V}}(M)$ to be the matrix acting on $V^{\otimes n}$ which acts on the subfactors $V_{i_1} \otimes V_{i_2} \dots$ as M , and acts trivially on any other factors. It will be convenient to use the R -index (or tensor space index) notation for actions on tensor space [26], in which, given \mathbf{V} , we write simply $M_{i_1 i_2 \dots i_m}$ for $\mathcal{I}_{i_1, i_2, \dots, i_m}^{\mathbf{V}}(M)$. Thus $\mathcal{R}_q(\mathcal{U}_i) = (U(q))_{i \ i+1}$ for general n , for the ordinary labelling scheme of $V^{\otimes n}$. However, after the ‘folding’ (34) of the open spin chain, the tensor factors have been relabelled, so the R -index notation there gives:

$$\mathcal{R}_{\hat{r}}(\mathcal{U}_{n+l}) = (U(\hat{r}))_{l^+ \ (l+1)^+}, \quad \mathcal{R}_r(\mathcal{U}_{n-l}) = (U(r))_{(l+1)^- \ l^-} = (U(r^{-1}))_{l^- \ (l+1)^-}, \quad (35)$$

so

$$\Theta(\mathcal{U}_l) = (U(r^{-1}))_{l^- \ (l+1)^-} \ (U(\hat{r}))_{l^+ \ (l+1)^+} \quad (36)$$

³For example, in case i the condition is $Q^2 = -\frac{\kappa + \delta_e q}{\kappa + \delta_e q^{-1}}$ and in case ii it is $Q^2 = -\frac{\kappa - i\delta_e}{\kappa + i\delta_e}$.

Notice from (26) that the single index l on \mathcal{U}_l is associated to a mirror image pair (l^-, l^+) in the underlying $V^{\otimes 2n}$ (a coupled pair in the folded scheme). Accordingly we introduce the space/mirror-space notation $\tilde{l} = (l^-, l^+)$. We correspondingly extend the tensor space index notation so that if an operator M acts on $V^{\otimes m}$ then $M_{\tilde{i}_1 \tilde{i}_2 \dots}$ acts on $V^{\otimes 2n}$ as $M_{i_1^- i_2^- \dots} M_{i_1^+ i_2^+ \dots}$. Thus $\mathcal{P}_{\tilde{k}\tilde{l}} = \mathcal{P}_{k-l^-} \mathcal{P}_{k+l^+}$. While if M acts on $(V \otimes V)^{\otimes m}$ then $M_{\tilde{i}_1 \tilde{i}_2 \dots}$ acts on $V^{\otimes 2n}$ by acting non-trivially on the appropriately positioned mirror image pairs: $(V_{i_1^-} \otimes V_{i_1^+}) \otimes (V_{i_2^-} \otimes V_{i_2^+}) \otimes \dots$

3.2 R -matrices and K -matrices for the gemini spin chain

The gemini spin chain is the solution to the Yang-Baxter equation (1) corresponding to using Θ in (8). The R -matrix can be written using the index notation as follows. Define $\check{U}_{kl}(r) = \mathcal{P}_{kl} (U(r))_{kl}$ and for $k \neq l \in \{1, 2, \dots, n\}$

$$R_{\tilde{k}\tilde{l}}(\lambda) = \mathcal{P}_{k-l^-} \mathcal{P}_{k+l^+} (\sinh \mu(\lambda + i) + \sinh \mu \lambda U_{k-l^-}(r^{-1}) U_{k+l^+}(\hat{r})) \quad (37)$$

$$= \sinh \mu(\lambda + i) \mathcal{P}_{k-l^-} \mathcal{P}_{k+l^+} + \sinh \mu \lambda \check{U}_{k-l^-}(r^{-1}) \check{U}_{k+l^+}(\hat{r}) \quad (38)$$

For $n = 2$ this is a 16×16 matrix, given explicitly in Appendix A. Define $\check{R}_{\tilde{k}\tilde{l}}(\lambda) = \mathcal{P}_{\tilde{k}\tilde{l}} R_{\tilde{k}\tilde{l}}(\lambda)$.

Note that the bulk space is significantly changed from that of the basic YBE solution for XXZ. Here the entire bulk space acquires a mirror image (a mirror copy V_{l^+} of each V_{l^-}). Neither side, viewed in isolation, retains the defining q -parameter in the usual way, nor do they have the same q -parameter as each other,⁴ and K acts on $V_{1^-} \otimes V_{1^+}$.

Proposition 5 *Representation Θ is a crossing representation, with $V = V_- \otimes V_+$ ($V_{\pm} \cong \mathbb{C}^2$).*

Proof: This R -matrix satisfies the unitarity and crossing properties in the form

$$R_{\tilde{k}\tilde{l}}(\lambda) R_{\tilde{l}\tilde{k}}(-\lambda) \propto \mathbb{I}, \quad R_{\tilde{k}\tilde{l}}(\lambda) = -\mathcal{V}_{\tilde{k}} R_{\tilde{l}\tilde{k}}^t(-\lambda - i) \mathcal{V}_{\tilde{k}}, \quad (39)$$

where

$$\mathcal{V}_{\tilde{k}} = \mathcal{V}_{k^-}^X(r^{-1}) \mathcal{V}_{k^+}^X(\hat{r}), \quad \text{and} \quad \mathcal{V}^X(p) = \begin{pmatrix} 0 & -ip^{-\frac{1}{2}} \\ ip^{\frac{1}{2}} & 0 \end{pmatrix}. \quad (40)$$

□

The K -matrix (9) in type I may be given in the 4×4 matrix form acting on $V_{1^-} \otimes V_{1^+}$ as $K(\lambda) = K^{(I)}(\lambda) = x(\lambda)\mathbb{I} + y(\lambda)M_{n=1}^I$:

$$K^{(i)}(\lambda) = \begin{pmatrix} x(\lambda) & & & \\ & w^+(\lambda) & f(\lambda) & \\ & f(\lambda) & w^-(\lambda) & \\ & & & x(\lambda) \end{pmatrix} \quad K^{(ii)}(\lambda) = \begin{pmatrix} w^+(\lambda) & & & f(\lambda) \\ & x(\lambda) & & \\ & & x(\lambda) & \\ f(\lambda) & & & w^-(\lambda) \end{pmatrix} \quad (41)$$

where

$$w^{\pm}(\lambda) = x(\lambda) + \frac{Q^{\pm 1} \delta_e}{Q + Q^{-1}} y(\lambda), \quad f(\lambda) = -\frac{\delta_e}{Q + Q^{-1}} y(\lambda) \quad (42)$$

⁴This system is also radically different from spin ladder systems such as in [36–38] — see [26] for an explicit comparison; and from systems obtained by fusion.

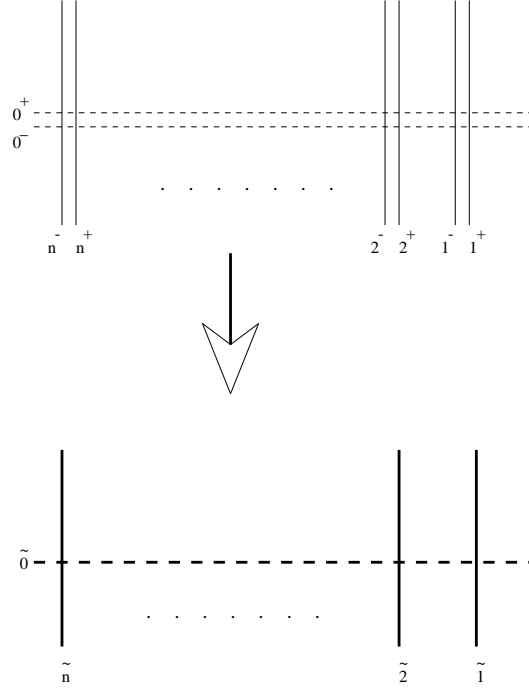


Figure 2: The monodromy matrix schematic.

(We omit the analogous K -matrix for types $(+)$, (iii) for brevity.) All the K -matrices satisfy unitarity, i.e.,

$$K(\lambda) K(-\lambda) \propto \mathbb{I}. \quad (43)$$

Recall that each spin in a spin chain may be thought of as having two legs ‘in’ and two legs out (the vertex model picture), and the factors V in tensor space are the configuration spaces of individual legs. Since our spins are doubled up (mirror image pairs) their legs are all composites of two simple legs. In the usual monodromy matrix formulation [2, 3] of a spin chain the legs in the lateral direction (within the transfer matrix layer) are all labelled 0, while the transverse legs are given the label of the corresponding spin. In our case the lateral direction legs are still composite legs, with one component leg coming from the ‘real’ and one from the ‘mirror’ side. Accordingly we will (ab)use the \tilde{l} notation, so that these legs are labelled $\tilde{0} = (0^-, 0^+)$ (see figure 2) when it is convenient to do so.

3.3 The algebraic Hamiltonian

Irrespective of whether the indices are composite or otherwise, the n -site chain *monodromy matrix* [2, 3] is, as usual,

$$T_0(\lambda) = R_{0n}(\lambda) R_{0n-1}(\lambda) \dots R_{01}(\lambda), \quad \hat{T}_0(\lambda) = R_{10}(\lambda) R_{20}(\lambda) \dots R_{n0}(\lambda) \quad (44)$$

Then by Theorem 1 and Proposition 3 of [12] the open spin chain *transfer matrix*

$$t(\lambda) = \text{tr}_0 M_0 K_0^+(\lambda) T_0(\lambda) K_0^-(\lambda) \hat{T}_0(\lambda), \quad (45)$$

obeys $[t(\lambda), t(\lambda')] = 0$ for all λ, λ' . Here K_0^\pm denotes the K -matrix for the left/right boundary of the chain (and M_0 is a physically unimportant correction to the right boundary term — see below). In what follows K^+ will be unit, K^- will be unit or given by K from (9) with ρ a crossing representation and $M_0 = \mathcal{V}_0^t \mathcal{V}_0$ (its explicit expression in case $\rho = \Theta$ is given in Appendix A).

Note that for $K^- = \mathbb{I}$ in particular the local R -matrix crossing condition (18) implies [39] that the open transfer matrix has a crossing symmetry:

$$t(\lambda) = t(-\lambda - i) \quad (46)$$

Given the set of commuting objects $t(\lambda)$, the *Hamiltonian* is (up to a choice of overall factor) $t'(0)$ (this commutes with $t(\lambda)$ by elementary considerations). Our choice of normalization is

$$\mathcal{H} = -\frac{\sinh^{1-2n}(i\mu)}{4\mu x(0)} \left(\text{tr}_0 M_0 \right)^{-1} t'(0) \quad (47)$$

Proposition 6 *If R , K are given as in (8), (9), and ρ is a crossing representation, then the Hamiltonian (47) may be written*

$$\mathcal{H} = \mathcal{H}^\rho = \frac{-1}{2} \sum_{i=1}^{n-1} \rho(\mathcal{U}_i) - \frac{\sinh(i\mu)y'(0)}{4\mu x(0)} \rho(\mathbf{e}) + \left(c - \frac{\sinh(i\mu)x'(0)}{4\mu x(0)} \right) \rho(1) \quad (48)$$

where $c = -\frac{n}{2} \cosh(\mu i) + \frac{1}{4 \cosh(\mu i)}$.

Proof: We have from (8), (9)

$$R_{kl}(0) = \sinh(\mu i) \mathcal{P}_{kl} \quad K^-(0) = K^{(I)}(0) = x(0) \mathbb{I}$$

so

$$\begin{aligned} t'(0) &= \text{tr}_0 \left(M_0 (\sinh(\mu i))^{2n-1} x(0) \left(\sum_{i=1}^n \mathcal{P}_{0n} \dots R'_{0i}(0) \dots \mathcal{P}_{01} \mathbb{I} \mathcal{P}_{10} \dots \mathcal{P}_{i0} \dots \mathcal{P}_{n0} \right. \right. \\ &\quad \left. \left. + \frac{\sinh(\mu i)}{x(0)} \mathcal{P}_{0n} \dots \mathcal{P}_{02} \mathcal{P}_{01} K^{-'}_0(0) \mathcal{P}_{10} \mathcal{P}_{20} \dots \mathcal{P}_{n0} + \sum_{i=1}^n \mathcal{P}_{0n} \dots \mathcal{P}_{0i} \dots \mathcal{P}_{01} \mathbb{I} \mathcal{P}_{10} \dots R'_{i0}(0) \dots \mathcal{P}_{n0} \right) \right) \\ &= \text{tr}_0 \left(M_0 (\sinh(\mu i))^{2n-1} x(0) \left(\sum_{i=1}^n \mathcal{P}_{0n} \dots \mathcal{P}_{0i+1} R'_{0i}(0) \mathcal{P}_{i0} \mathcal{P}_{i+10} \dots \mathcal{P}_{n0} \right. \right. \\ &\quad \left. \left. + \frac{\sinh(\mu i)}{x(0)} \mathcal{P}_{01} K^{-'}_0(0) \mathcal{P}_{10} + \sum_{i=1}^n \mathcal{P}_{0n} \dots \mathcal{P}_{0i} R'_{i0}(0) \dots \mathcal{P}_{n0} \right) \right) \\ &= \text{tr}_0 \left(M_0 (\sinh(\mu i))^{2n-1} x(0) \left(\frac{\sinh(\mu i)}{x(0)} K^{-'}_1(0) + 2 \sum_{i=1}^{n-1} \mathcal{P}_{i+1i} R'_{ii+1}(0) + 2 \mathcal{P}_{0n} R'_{n0}(0) \right) \right) \end{aligned}$$

$$= (\sinh(\mu i))^{2n-1} x(0) \left(\text{tr}_0(M_0) \left(\frac{\sinh(\mu i)}{x(0)} K^{-'}_1(0) + 2 \sum_{i=1}^{n-1} \mathcal{P}_{i+1i} R'_{ii+1}(0) \right) + \text{tr}_0(M_0 (2\mathcal{P}_{0n} R'_{n0}(0))) \right)$$

Defining

$$\mathcal{H}_{kl} = -\frac{1}{2\mu} \left(\frac{d}{d\lambda} \mathcal{P}_{kl} R_{kl}(\lambda) \right) \Big|_{\lambda=0} \quad (49)$$

we have

$$\mathcal{H} = \sum_{l=1}^{n-1} \mathcal{H}_{ll+1} - \frac{\sinh(i\mu)}{4\mu x(0)} \left(\frac{d}{d\lambda} K_1^-(\lambda) \right) \Big|_{\lambda=0} + \frac{\text{tr}_0 M_0 \mathcal{H}_{n0}}{\text{tr}_0 M_0} \quad (50)$$

It follows immediately from (8) and (49) that

$$\mathcal{H}_{ll+1} = -\frac{1}{2} (\rho(\mathcal{U}_l) + \cosh(i\mu) \rho(1)) \quad (51)$$

and by (9), $K'(0) = x'(0) \rho(1) + y'(0) \rho(e)$. Finally by the crossing assumption

$$\begin{aligned} \text{tr}_0(M_0 \mathcal{H}_{n0}) &= \frac{-1}{2} \text{tr}_0(M_0 (\cosh(i\mu) + \mathcal{V}_0 \mathcal{P}_{n0}{}^{t_0} \mathcal{V}_n)) = \frac{-1}{2} (\cosh(i\mu) \text{tr}_0(M_0) + \text{tr}_0(M_0 \mathcal{V}_0 \mathcal{P}_{n0}{}^{t_0} \mathcal{V}_n)) \\ &\stackrel{\text{prop.2}}{=} \frac{-1}{2} (-2 \cosh^2(i\mu) \mathbb{I} + \mathcal{V}_n^t \mathcal{V}_n^t \mathcal{V}_n \text{tr}_0(\mathcal{P}_{n0}{}^{t_0} \mathcal{V}_n)) \stackrel{\text{prop.1}}{=} \frac{-1}{2} (-2 \cosh^2(i\mu) \mathbb{I} + \mathcal{V}_n^t \mathcal{V}_n^t \mathcal{V}_n \mathcal{V}_n) \end{aligned}$$

But $\mathcal{V}_n^t \mathcal{V}_n^t \mathcal{V}_n \mathcal{V}_n = \mathbb{I}$ again by assumption, so we are done.

4 Quantum group symmetry

At its most general, the notion of ‘spin chain Hamiltonian’ means, amongst other things, a *sequence* of matrices indexed by \mathbb{N} , such that the n^{th} matrix acts on $V^{\otimes n}$ for some V (of course only very special kinds of matrices are allowed, but this structure will be sufficient for now). Note that the XXZ Hamiltonian is such a sequence:

$$\left\{ \mathcal{H}_{XXZ} \propto \sum_{i=1}^{n-1} \mathcal{R}_q(\mathcal{U}_i) = \sum_{i=1}^{n-1} (U(q))_{i \ i+1} \mid n = 2, 3, \dots \right\}. \quad (52)$$

Let \mathcal{G} be a quantum group [40] with coproduct $\Delta : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G}$. Given a representation of $\mathcal{G} \otimes \mathcal{G}$ one may use the coproduct to give an action of \mathcal{G} . Thus in particular there is a notion of tensor product of any two representations of \mathcal{G} (just as there is for ordinary groups). By a \mathcal{G} -symmetry of the Hamiltonian we mean an action P of \mathcal{G} on V , such that the n -fold product of P on $V^{\otimes n}$ commutes with the n^{th} \mathcal{H} matrix.

In this section we consider the symmetry of the Hamiltonian \mathcal{H} from (48) in cases $\rho = \Theta^I$. (Note that at very least a \mathcal{G} -symmetry implies some sort of global limit structure on the Hamiltonian sequence. Our representation theoretic construction will be used to exhibit such a structure in section 4.5.) First, it is useful to recall some basic definitions and results.

If \mathcal{G} is a bialgebra with coproduct Δ then set $\Delta^2 = \Delta$ and then, for $n > 2$, $\Delta^n = (\Delta \otimes 1 \otimes 1 \otimes \dots \otimes 1) \Delta^{n-1}$ [40]. The following result is very familiar — we state it in a slightly unusual form suitable for use here.

Proposition 7 Fix a space V and a biaglebra \mathcal{G} with generators $\{\mathcal{E}_\alpha, \mathcal{F}_\alpha, \mathcal{K}_\alpha, \mathcal{K}_\alpha^{-1}\}_{\alpha \in I}$ (I some index set) and coproduct

$$\begin{aligned}\Delta(\mathcal{E}_\alpha) &= \mathcal{K}_\alpha^{-1} \otimes \mathcal{E}_\alpha + \mathcal{E}_\alpha \otimes \mathcal{K}_\alpha, \\ \Delta(\mathcal{F}_\alpha) &= \mathcal{K}_\alpha^{-1} \otimes \mathcal{F}_\alpha + \mathcal{F}_\alpha \otimes \mathcal{K}_\alpha, \\ \Delta(\mathcal{K}_\alpha^{\pm 1}) &= \mathcal{K}_\alpha^{\pm 1} \otimes \mathcal{K}_\alpha^{\pm 1}.\end{aligned}\tag{53}$$

Suppose that $\rho : \mathcal{G} \rightarrow \text{End}(V)$ a representation, so that $\rho^{\otimes 2}(\Delta(-))$ a representation on $V_1 \otimes V_2$. Now let $M \in \text{End}(V_1 \otimes V_2)$ and $K \in \text{End}(V_1)$ be matrices such that

$$[\rho^{\otimes 2}(\Delta(x)), M] = 0, \quad [\rho(x), K] = 0 \quad \forall x \in S = \{\mathcal{E}_\alpha, \mathcal{K}_\alpha, \mathcal{K}_\alpha^{-1}\}.\tag{54}$$

Then with $M_{i+1} = \mathcal{I}_{i+1}^V(M)$ and $K_i = \mathcal{I}_i^V(K)$

$$[\rho^{\otimes n}(\Delta^n(x)), M_{i+1}] = 0, \quad [\rho^{\otimes n}(\Delta^n(x)), K_i] = 0\tag{55}$$

for all $n \geq 2$, all $x \in S$, and all $i = 1, 2, \dots, n-1$.

Proof: Firstly, noting that $\rho^{\otimes n}(\Delta^n(\mathcal{K}_\alpha)) = (\rho(\mathcal{K}_\alpha))^{\otimes n}$, for any given i (55) _{$x=\mathcal{K}_\alpha^{\pm 1}$} is a direct consequence of the form of M_{i+1} (respectively K_i) and (54) _{$x=\mathcal{K}_\alpha^{\pm 1}$} . Now define

$$\chi_j = \rho^{\otimes n}(\mathcal{K}_\alpha^{-1} \otimes \dots \otimes \mathcal{K}_\alpha^{-1} \otimes \underbrace{\mathcal{E}_\alpha}_{j\text{-th position}} \otimes \mathcal{K}_\alpha \dots \otimes \mathcal{K}_\alpha)$$

so that $\rho^{\otimes n}(\Delta^n(\mathcal{E}_\alpha)) = \sum_j \chi_j$. If $i > j$ or $i+1 < j$, $[\chi_j, M_{i+1}] = 0$ by (54) _{$x=\mathcal{K}_\alpha^{\pm 1}$} , while $[\chi_i + \chi_{i+1}, M_{i+1}] = 0$ by (54) _{$x=\mathcal{E}_\alpha$} , and so on. Similarly $[\chi_j, K_i] = 0$. \square

Comparing Proposition 7 and (52) we see, that for Hamiltonians of this form, verification of \mathcal{G} -symmetry may be possible by strictly local calculations.

4.1 Example: XXZ and the quantum group $U_q(sl(2))$

Definition 3 [41, 42] For $q \in \mathbb{C} \setminus \{0, 1, -1\}$ let $\mathcal{G} = \mathcal{G}_q := U_q(sl(2))$, the algebra with generators \mathcal{E}, \mathcal{F} and \mathcal{K} and relations

$$[\mathcal{E}, \mathcal{F}] = \frac{\mathcal{K}^2 - \mathcal{K}^{-2}}{q - q^{-1}}, \quad \mathcal{K} \mathcal{E} = q \mathcal{E} \mathcal{K}, \quad \mathcal{K} \mathcal{F} = q^{-1} \mathcal{F} \mathcal{K}.\tag{56}$$

Lemma 1 There is an algebra automorphism w on \mathcal{G} given by $w(\mathcal{E}) = \mathcal{F}$, $w(\mathcal{F}) = \mathcal{E}$, $w(\mathcal{K}) = \mathcal{K}^{-1}$ (the ‘Cartan’ automorphism).

Recall that the standard equivalence classes of generically simple modules of \mathcal{G} may be indexed by the non-negative integers [43], with the module indexed by ν having dimension $\nu+1$. In particular we have a $\nu=0$ action on \mathbb{C} given by:

$$\epsilon(\mathcal{E}) = \epsilon(\mathcal{F}) = 0, \quad \epsilon(\mathcal{K}^{\pm 1}) = 1\tag{57}$$

and a $\nu = 1$ action on \mathbb{C}^2 (with basis $\{v_1, v_2\}$) written in terms of Pauli matrices as

$$\rho(\mathcal{K}) = q^{\frac{1}{2}\sigma^z} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad \rho(\mathcal{E}) = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(\mathcal{F}) = \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (58)$$

A simple calculation⁵ shows that the $T_n(q)$ generators in the XXZ representation (22) commute with the $\rho^{\otimes n}$ action of $U_q(sl(2))$ on $(\mathbb{C}^2)^{\otimes n}$ for $n = 2$ and hence, via Proposition 7, any n :

$$\left[\rho^{\otimes n}(\Delta^n(x)), \mathcal{R}_q(\mathcal{U}_i) \right] = \left[\rho^{\otimes n}(\Delta^n(x)), (U(q))_{i \ i+1} \right] = 0 \quad \forall x \in \mathcal{G} \quad (59)$$

That is, we have the following extraordinary (but very well known) result.

The action \mathcal{R}_q of $T_n(q)$ on $End((\mathbb{C}^2)^{\otimes n})$ commutes with the action $\rho^n(\Delta^n)$ of \mathcal{G} , for any n [44]. Indeed [45]

$$T_n(q) \cong \text{End}_{U_q(sl(2))}((\mathbb{C}^2)^{\otimes n}) \quad (60)$$

Applying this to \mathcal{H} in (52) we have the $U_q(sl(2))$ symmetry of the Hamiltonian.

For reference in the *unfamiliar* generalization we are about to explore, let us unpack the familiar duality (60) a little. For each choice of q , on the one side we have a tower of algebras, and on the other a single algebra, but equipped with coproduct, so that each has a natural action on $V^{\otimes n}$ for each n . Each action lies in the commutator of the other. In fact each action *is* the commutator of the other [45].

The questions raised are: What is the commutator of the Θ action of $T_n(q)$ on $V^{\otimes 2n}$? Can this commutator be constructed in an analogously n -independent way? I.e., can it be constructed as the iterated coproduct of a bialgebra action? This would be the requirement for identifying the commutant as a quantum group symmetry of any associated spin chain. These are ‘big’ questions. In section 4.5 we answer the first one at least formally, by determining the structure of the commutator and showing that it is appropriately n -stable. Next, however, we answer affirmatively the slightly less ambitious question: Can any nontrivial part of the commutator of b_n on $V^{\otimes 2n}$ be constructed as an iterated coproduct?

4.2 $U_q(sl(2))$ actions on $V \otimes V$

By the general theory above, one may begin the search for \mathcal{G} -symmetries of $\mathcal{H} = \mathcal{H}^\Theta$ by looking at the ways in which $V \otimes V$ can be equipped with the property of \mathcal{G} -module. We will think, in particular, of $\sigma(\mathcal{G})$ acting on $V_{1-} \otimes V_{1+}$, and $\sigma^{\otimes 2}(\Delta(\mathcal{G}))$ acting on $(V_{1-} \otimes V_{1+}) \otimes (V_{2-} \otimes V_{2+})$, and so on.⁶ Here we report on $U_q(sl(2))$ symmetries we have found (although we also show that this does not rule out other quantum groups). In terms of equivalence classes of actions we can essentially

⁵It should be noted that since we have given the representation of T_n in an explicit matrix form, there must be a consistent convention for the rendering of direct products into matrix form. Here we use the ‘Greek’ convention:

$$(a, b) \otimes (c, d) = (ac, ad, bc, bd).$$

⁶NB, These details are arbitrary for \mathcal{G} itself, but we will be composing with a specific action of b_n , so it is necessary to be specific on this side also.

characterize any such module by its irreducible content; and as already mentioned the $U_q(sl(2))$ irreducibles are indexed by the whole numbers. Since $V \otimes V$ is a four dimensional space, we have classes of modules of type $1+1+1+1$, $1+1+2$, $2+2$, $1+3$, 4 . The first of these is $\epsilon \oplus \epsilon \oplus \epsilon \oplus \epsilon$, which is trivial. For the others, of course, one is looking for commutation with a *specific* action of b_n , so one must check commutation for specific actions. Since there are infinitely many of these in each case a certain amount of sensible guesswork is required. Further, it is not clear that the parameter q in $U_q(sl(2))$ need be the same as the parameter q in b_n , and we will not make this assumption!

For example, restricting to the case of *trivial* boundaries, one notes that the ‘real’ and ‘mirror’ sides of the construction of Θ are, in a suitable sense, decoupled. Thus there is an obvious pair of actions in the $\rho \oplus \rho$ equivalence class, as follows.

4.3 A vestigial symmetry of the asymmetric chain with $K^- \propto \mathbb{I}$

Consider representations of $U_r(sl(2))$ and $U_{\hat{r}}(sl(2))$ respectively on $V_{1-} \otimes V_{1+} = \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\rho_1(-) = \rho_{q=r}(w(-)) \otimes \mathbb{I}, \quad \rho_2(-) = \mathbb{I} \otimes \rho_{q=\hat{r}}(-) \quad (61)$$

NB, these are not tensor products of representations. Rather, regarded as an action on $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$, each is simply isomorphic to $\rho \oplus \rho$ (for the appropriate choice of q -parameter). Note that ρ_1 acts non-trivially on the ‘real’ (left) space only, whereas ρ_2 acts non-trivially on the mirror (right) space. One is then to understand in the representation $\rho_1^{(2)}(-)$ on $(\mathbb{C}^2 \times \mathbb{C}^2)^{\otimes 2}$ given by

$$\rho_1^{(n)}(-) = \rho_r^{\otimes n}(\Delta^n(w(-))) \otimes \mathbb{I}^n \quad \rho_2^{(n)}(-) = \mathbb{I}^n \otimes \rho_{\hat{r}}^{\otimes n}(\Delta^n(-))$$

that the non-trivial factor acts on $V_{2-} \otimes V_{1-}$ and the trivial one on $V_{1+} \otimes V_{2+}$ (and complementarily for ρ_2). So by (59) (or otherwise trivially)

$$\left[\rho_l^{(2)}(x), (U(r^{-1}))_{1- \ 2-} \right] = 0, \quad \left[\rho_l^{(2)}(x), (U(\hat{r}))_{1+ \ 2+} \right] = 0, \quad (62)$$

for $l \in \{1, 2\}$, $x \in \mathcal{G}$, so by (36)

$$\left[\rho_l^{(2)}(x), \Theta(\mathcal{U}_1) \right] = 0, \quad l \in \{1, 2\} \quad (63)$$

By Proposition 7, relations (63) imply corresponding statements for any number of sites n . Recalling also (48) it follows (at least for $K^- = \mathbb{I}$) that

$$\left[\rho_l^{(n)}(x), \mathcal{H} \right] = 0. \quad (64)$$

Since the two quantum group actions commute with *each other* trivially, we conclude that the model for $K^- = \mathbb{I}$ has a $U_r(sl(2)) \otimes U_{\hat{r}}(sl(2))$ symmetry.

This exercise serves to illustrate the point that the close natural relationships between spin-chain symmetry and quantum group, which we take for granted in conventional spin chains, is not necessarily obvious in general.

The K -matrices (41) do not commute with the ρ_l actions (61), so a nontrivial boundary term in the Hamiltonian breaks the $U_r(sl(2)) \otimes U_{\hat{r}}(sl(2))$ symmetry.

4.4 Boundary stable symmetries

Let e_{ij} be the 4×4 elementary matrix: $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$ and

$$\begin{aligned} \tilde{S}^+ &= e_{14} = \sigma^+ \otimes \sigma^+ & \tilde{S}^- &= e_{41} = \sigma^- \otimes \sigma^- & \tilde{S}^z &= \frac{1}{2}(e_{11} - e_{44}) \\ S^+ &= e_{23} = \sigma^+ \otimes \sigma^- & S^- &= e_{32} & S^z &= \frac{1}{2}(e_{22} - e_{33}) \end{aligned} \quad (65)$$

For any \tilde{q} , there is a representation $\sigma : U_{\tilde{q}}(sl(2)) \rightarrow \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, given by

$$\sigma(\mathcal{K}) = \tilde{q}^{-S^z}, \quad \sigma(\mathcal{F}) = S^+, \quad \sigma(\mathcal{E}) = S^- \quad (66)$$

(Note that $\sigma \cong \epsilon \oplus \rho \oplus \epsilon$, but that it acts non-trivially on both the ‘real’ and ‘mirror’ spaces.) In order to apply proposition 7 to commutation with Θ we need to express $\Theta(\mathcal{U}_i)$ in the form $M_{i \ i+1}$. Our convention for the explicit action of $\sigma^{\otimes n}(\Delta^n)$ was chosen to make this possible. The explicit matrix form of $\Theta(\mathcal{U}_i)$ is the permutation of (27) given in Appendix A. Direct calculation at $n = 2$ then shows that $[\sigma^{\otimes 2}(\Delta(\mathcal{K})), \Theta(\mathcal{U}_1)] = 0$, and

$$\begin{aligned} & [\sigma^{\otimes 2}(\Delta(\mathcal{F})), \Theta(\mathcal{U}_1)]_{kl} = -[\sigma^{\otimes 2}(\Delta(\mathcal{E})), \Theta(\mathcal{U}_1)]_{lk} = \\ & \tilde{q}^{-\frac{1}{2}}(\tilde{q} - q) \left(\delta_{k6}(-r^{-1}\delta_{l4} - \hat{r}^{-1}\delta_{l13} + \delta_{l10} - q^{-1}\delta_{l7}) - \delta_{l11}(-q^{-1}\delta_{k7} - r^{-1}\delta_{k4} - \hat{r}^{-1}\delta_{k13} + \delta_{k10}) \right) \end{aligned} \quad (67)$$

(here we use the flattened labels $k, l \in \{1, 2, \dots, 16\}$). In summary

$$[\sigma^{\otimes 2}(\Delta(x)), \Theta(\mathcal{U}_1)] = 0, \quad \forall x \in \mathcal{G}_{\tilde{q}} \quad \Longleftrightarrow \quad \tilde{q} = q \quad (68)$$

Direct calculation at $n = 1$ shows (for any \tilde{q}) that

$$[\sigma(x), \Theta^{ii}(\mathbf{e})] = [\sigma(x), K^{(ii)}(\lambda)] = 0 \quad \forall x \in \mathcal{G}_{\tilde{q}} \quad (69)$$

but that there is no such commutation for Θ^i . By Proposition 7, relations (68), (69) imply

Proposition 8 *For every n , $[\Theta(\mathcal{U}_i), \sigma^{\otimes n}(\Delta^n(x))] = [\Theta^{ii}(\mathbf{e}), \sigma^{\otimes n}(\Delta^n(x))] = 0$, and hence the Hamiltonian $\mathcal{H} = \mathcal{H}^{\Theta^I}$ (48) with boundary term of type (ii) or (0) has a $U_q(sl(2))$ symmetry:*

$$[\mathcal{H}, \sigma^{\otimes n}(\Delta^n(x))] = 0. \quad (70)$$

Another representation $\pi : U_{\tilde{q}}(sl(2)) \rightarrow \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is

$$\pi(\mathcal{K}) = \tilde{q}^{\tilde{S}^z}, \quad \pi(\mathcal{E}) = \tilde{S}^+, \quad \pi(\mathcal{F}) = \tilde{S}^-. \quad (71)$$

One can show directly that $[\pi^{\otimes 2}(\Delta(\mathcal{K})), \Theta(\mathcal{U}_1)] = 0$, and

$$\begin{aligned} & [\pi^{\otimes 2}(\Delta(\mathcal{F})), \Theta(\mathcal{U}_1)]_{kl} = -[\pi^{\otimes 2}(\Delta(\mathcal{E})), \Theta(\mathcal{U}_1)]_{lk} = \\ & \tilde{q}^{-\frac{1}{2}}(\tilde{q} - \sqrt{-1}) \left(\delta_{k16}(\delta_{l4} - \hat{r}^{-1}\delta_{l7} - r\delta_{l10} + i\delta_{l13}) - \delta_{l1}(\delta_{k4} - \hat{r}^{-1}\delta_{k7} - r\delta_{k10} + i\delta_{k13}) \right) \end{aligned} \quad (72)$$

In summary

$$\left[\pi^{\otimes 2}(\Delta(x)), \Theta(\mathcal{U}_1) \right] = 0 \quad \forall x \in \mathcal{G}_{\tilde{q}} \quad \Longleftrightarrow \quad \tilde{q} = \sqrt{-1} \quad (73)$$

Direct calculation shows (see (41) and (71)) that

$$\left[\pi(x), \Theta^i(\mathbf{e}) \right] = \left[\pi(x), K^{(i)}(\lambda) \right] = 0 \quad \forall x \in \mathcal{G}_{\tilde{q}}. \quad (74)$$

From (73) and (74) and Proposition 7 we have

Proposition 9 *For every n , the Hamiltonian $\mathcal{H} = \mathcal{H}^{\Theta^I}$ (48) with boundary term of type (i) or (0) has a $U_{\sqrt{-1}}(sl(2))$ symmetry:*

$$\left[\mathcal{H}, \pi^{\otimes n}(\Delta^n(x)) \right] = 0. \quad (75)$$

In summary: we have exposed a $U_q(sl(2)) \otimes U_i(sl(2))$ symmetry of the trivial boundary Hamiltonian. The $U_q(sl(2))$ symmetry is preserved in the presence of the nontrivial boundary (ii), but broken by the boundary (i); whereas the $U_i(sl(2))$ symmetry is preserved in the presence of (i), but broken by (ii). (For comparison, the Hamiltonians considered here are written out in terms of Pauli matrices in Appendix D.)

While it is possible that the symmetry described here constitutes the full \mathcal{G} -symmetry of \mathcal{H} in each boundary type, it is *not* the full commutator as it is in the XXZ case. To see this is a manageable exercise, comparing the generic simple multiplicities in Θ (given in [30] in some cases) with the corresponding data for the quantum group, as we will now show.

Non-trivial boundaries such as ours sometimes allow symmetries called boundary quantum algebras [22, 46–48]. These will be discussed in Section 5.

4.5 Representation theory of $\text{End}_{b_n}(V^{\otimes 2n})$

By (48) the spectrum of \mathcal{H} is determined (in principle) by the representation theory of T_n and b_n . In practice this theory is not yet an effective substitute for Bethe ansatz (cf. [49]), but some useful results may be gleaned by using the approaches in tandem. For example, Martin and Saleur’s extension of the representation \mathcal{R}_q to b_n [28] defines the spin-1/2 XXZ chain with non-diagonal boundaries.⁷ By our general theory, therefore, this chain has the same spectrum, up to multiplicities, as the gemini chain, if and only if the underlying representations have the same irreducible content up to multiplicities. We may derive the irreducible content of the Θ representations (and other tensor space representations such as \mathcal{R}_q) as follows.

Let A, B be algebras. An idempotent $e \in A$ is called a *localisation idempotent* from A to B if $eAe \cong B$. Through this isomorphism one may regard eA as a left B right A bimodule. Thus we have a functor $L : A\text{-mod} \rightarrow B\text{-mod}$ given by $L(M) = eM$ (here $A\text{-mod}$ denotes the category of left A -modules); and a functor $G : B\text{-mod} \rightarrow A\text{-mod}$ given, via the isomorphism,

⁷Apart from the significant disadvantage of not having reference states, this is the obvious candidate for studying non-trivial boundaries, and has received significant attention — see [29] for references.

by $G(N) = Ae \otimes_B N$. Thus, presence of a localisation idempotent ensures that there is a full embedding of the category of left (or right) B -modules in the corresponding category of A -modules (the functor L is called localisation, and is exact; and G , globalisation, is right exact). For example, appropriately normalised, \mathcal{U}_{n-1} is a localisation idempotent from T_n to T_{n-2} , and also from b_n to b_{n-2} . Both of these collections of algebras have *standard modules* (generically irreducible modules) Δ_ν^n , with the following properties.

Proposition 10 *The standard modules of T_n are indexed by $\nu \in \{n, n-2, \dots, 1/0\}$. The standard modules of b_n are indexed by $\lambda \in \{-n, -(n-2), \dots, (n-2), n\}$. Then*

- (i) *localisation takes a standard module to a standard module $\Delta_\nu^n \mapsto \Delta_\nu^{n-2}$, or zero if $|\lambda| > n-2$.*
- (ii) *localisation takes $\Theta : b_n$ to $\Theta : b_{n-2}$ (any I).*
- (ii') *localisation takes $\mathcal{R}_q : b_n$ to $\mathcal{R}_q : b_{n-2}$.*
- (iii) *restriction of standard modules is given by $\Delta_\nu^n \mapsto \Delta_{\nu+1}^{n-1} + \Delta_{\nu-1}^{n-1}$,*
- (iv) *restriction takes $\Theta : b_n \mapsto 4(\Theta : b_{n-1})$ (any I).*
- (iv') *restriction takes $\mathcal{R}_q : b_n \mapsto 2(\mathcal{R}_q : b_{n-1})$.*

Proof: The index theorems and (i) are standard results [30]; (ii) follows from equation(27); (iii) is again well known; (iv) follows from a simple calculation. \square

For M a module with a well defined filtration by standard modules (such as Θ), let us write $(M : \Delta_\nu)$ for the multiplicity of Δ_ν in M . By (i) and (ii) we have

Corollary 10.1 *The multiplicity $(\Theta : \Delta_\nu^n)$ does not depend on n , once $n \geq |\nu|$. (Similarly for \mathcal{R}_q .)*

By 10.1, (iii) and (iv) we have

Corollary 10.2 *Let \mathbb{C}^∞ have basis $\{e^\nu : \nu \in \mathbb{Z}\}$. Setting $\mathbb{C}^\infty \ni v = \sum_\nu (\Theta : \Delta_\nu^n) e^\nu$ and operator $\chi e^\nu = e^{\nu+1} + e^{\nu-1}$ then $\chi v = 4v$.*

Given the multiplicities for $\nu < n$, it follows that the only question at level n is the multiplicities of the one-dimensional modules $\Delta_{\pm n}^n$. The combined multiplicity is fixed by the dimension of Θ (for example in the b_n case):

$$(\Theta : \Delta_n^n) + (\Theta : \Delta_{-n}^n) = \dim(\Theta) - \sum_{|\nu| < n} \dim(\Delta_\nu^n) (\Theta : \Delta_\nu^n);$$

and the individual multiplicities by (iii) and (iv).

It follows that the multiplicities for *all* n are determined by the case $n = 1$ and either the case $n = 2$ or $n = 0$ (if this can be appropriately defined — in our case it denotes a copy of the ground ring). This proves proposition 3. The formal case $n = 0$ does not depend on the extension to Θ . By construction Θ^i and Θ^{ii} have the same multiplicities at $n = 1$, so we deduce that they are generically isomorphic. Thus the corresponding spin chains will have the same spectrum. (As we will see later, this abstract proof does not correspond to a straightforward connection at the level of Bethe Ansatz.)

Since the dimensions of the standard modules, and of Θ , are readily computed, the spectrum may be determined explicitly.

$$\begin{array}{cccccc}
\nu : & 0 & 1 & 2 & 3 & 4 \\
(\Theta : \Delta_\nu^{odd}) : & & 4 & & 56 & \cdots 780 \\
(\Theta : \Delta_\nu^{even}) : & 1 & & 15 & & 209 \\
n = 1 & & 1 & & & \\
n = 2 & 1 & & 1 & & \\
n = 3 & & 2 & & 1 & \\
n = 4 & 2 & & 3 & & 1
\end{array}$$

In case Θ^i (or Θ^{ii}) the corresponding picture is:

$$\begin{array}{cccccccc}
\nu : & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
(\Theta^I : \Delta_\nu^{odd}) : & 11 & & 1 & & 3 & & 41 & \\
(\Theta^I : \Delta_\nu^{even}) : & & 3 & & 1 & & 11 & & 153 \\
\dim(\Delta_\nu^n) & n=0 & & & & 1 & & & \\
& n=1 & & & 1 & & 1 & & \\
& n=2 & & 1 & & 2 & & 1 & \\
& & 1 & & 3 & & 3 & & 1
\end{array}$$

⁸The multiplicity combinatorics may be encoded in other ways too. Consider

$$\underbrace{\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & \ddots & \ddots \end{pmatrix}}_{\chi^+} \begin{pmatrix} 1 \\ 4 \\ 15 \\ 56 \\ \vdots \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 4 \\ 15 \\ 56 \\ \vdots \end{pmatrix}$$

22

And so on.

In case Θ^+ (resp. \mathcal{R}_q of [28]) the corresponding picture is:

$$\begin{array}{cccccccc}
\nu : & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
(\Theta^+ : \Delta_\nu^{odd}) : & 26 & & 2 & & 2 & & 26 & \\
(\Theta^+ : \Delta_\nu^{even}) : & & 7 & & 1 & & 7 & & 97 \\
(\mathcal{R}_q : \Delta_\nu^{odd}) : & 1 & & 1 & & 1 & & 1 & \\
(\mathcal{R}_q : \Delta_\nu^{even}) : & & 1 & & 1 & & 1 & & 1
\end{array}$$

which shows that there is not a unique groundstate for odd n in the Θ^+ case.

Comparing the multiplicities in the Θ cases with the actions of the quantum groups described above we see that the naive commutant is much bigger than the set of commuting matrices given by these actions. In section 5 we discuss quantum group actions which do not lie in the commutant themselves, but will underly the exposition of further symmetries, and also help us prepare to address the technical question of Bethe Ansatz asymptotics. On the other hand, since all multiplicities here are nonzero we have proved

Proposition 11 *The spectrum of gemini and boundary XXZ are the same up to multiplicities.*

5 Further symmetries

In this section we investigate further the symmetry of the open Hamiltonian \mathcal{H}^Θ in the presence of non-trivial boundaries (41). We will first need some notation.

5.1 The quantum affine algebra $U_q(\widehat{sl(2)})$

The Cartan matrix of the affine Lie algebra $\widehat{sl(2)}$ [50] is

$$(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad (76)$$

The quantum Kac-Moody algebra $U_q(\widehat{sl(2)})$ has Chevalley-Serre generators [41, 42, 51] e_i, f_i, k_i , $i = 1, 2$ with defining relations

$$\begin{aligned}
k_i k_j &= k_j k_i, & k_i e_j &= q^{\frac{1}{2}a_{ij}} e_j k_i, & k_i f_j &= q^{-\frac{1}{2}a_{ij}} f_j k_i, \\
[e_i, f_j] &= \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}}, & & & &
\end{aligned} \quad (77)$$

(and the q deformed Serre relations)

$$\begin{aligned}
e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i^2 - e_j e_i^3 &= 0 \\
f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 &= 0, & i &\neq j.
\end{aligned} \quad (78)$$

Note that if ϕ is a representation of $U_q(\widehat{sl(2)})$ then the map ϕ_0 on generators given by $\phi_0(e_1) = \phi(\mathcal{E})$, $\phi_0(f_1) = \phi(\mathcal{F})$, $\phi_0(k_1) = \phi(\mathcal{K})$, $\phi_0(e_2) = \phi(w(\mathcal{E}))$, $\phi_0(f_2) = \phi(w(\mathcal{F}))$, $\phi_0(k_2) = \phi(w(\mathcal{K}))$ satisfies (77). In particular if $\phi(\mathcal{E}^2) = 0$ (i.e. ϕ is a sum of representations of class ϵ and ρ) then ϕ_0 extends to a representation of $U_q(\widehat{sl(2)})$.

Note also that the relations (77), (78) are, except for the $i = j$ case of the commutator, homogeneous in each generator. It follows that given a representation, there is another with $\psi(e_i) \rightarrow c\psi(e_i)$, $\psi(f_i) \rightarrow c^{-1}\psi(f_i)$, for either i and any constant $c \neq 0$. In particular, for λ a scalar, there is a representation $\rho_\lambda : U_q(\widehat{sl(2)}) \rightarrow \text{End}(\mathbb{C}^2)$ given by

$$\begin{aligned} \rho_\lambda(e_1) &= \sigma^\pm, & \rho_\lambda(f_1) &= \sigma^\mp, & \rho_\lambda(k_1) &= q^{\pm\sigma^z} \\ \rho_\lambda(e_2) &= e^{-2\mu\lambda}\sigma^\mp, & \rho_\lambda(f_2) &= e^{2\mu\lambda}\sigma^\pm, & \rho_\lambda(k_2) &= q^{\mp\sigma^z}. \end{aligned} \quad (79)$$

This is the *evaluation representation* [41]. If ϕ above obeys $\phi(\mathcal{E}^2) = 0$ we call ϕ *linear*. If ϕ is linear the corresponding construction $\phi \rightarrow \phi_\lambda$ gives a representation of $U_q(\widehat{sl(2)})$.

Set $\mathcal{A} = U_q(\widehat{sl(2)})$. A Hopf algebra structure is defined on \mathcal{A} by introducing the coproduct⁹ $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$:

$$\begin{aligned} \Delta(e_i) &= k_i^{-1} \otimes e_i + e_i \otimes k_i \\ \Delta(f_i) &= k_i^{-1} \otimes f_i + f_i \otimes k_i \\ \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}. \end{aligned} \quad (80)$$

5.2 Boundary quantum algebra symmetry

We now construct, cf. [22, 47, 48], realizations of the so called boundary quantum algebra generators which commute with \mathcal{H}^{Θ^f} . We shall need versions of the evaluation representation (79) applied to (61), (71), (66): $\pi \rightarrow \pi_\lambda$, $\sigma \rightarrow \sigma_\lambda$, $\rho_1 \rightarrow \rho_\lambda^1$, $\rho_2 \rightarrow \rho_\lambda^2$. E.g. $\sigma_\lambda : U_q(\widehat{sl(2)}) \rightarrow \text{End}(\mathbb{C}^4)$:

$$\begin{aligned} \sigma_\lambda(k_1) &= q^{-S^z}, & \sigma_\lambda(e_1) &= S^-, & \sigma_\lambda(f_1) &= S^+ \\ \sigma_\lambda(k_2) &= q^{S^z}, & \sigma_\lambda(e_2) &= e^{-2\mu\lambda}S^+, & \sigma_\lambda(f_2) &= e^{2\mu\lambda}S^- \end{aligned} \quad (81)$$

Similarly, let $\pi_\lambda : U_q(\widehat{sl(2)}) \rightarrow \text{End}(\mathbb{C}^4)$ such that

$$\begin{aligned} \pi_\lambda(k_1) &= i^{\tilde{S}^z}, & \pi_\lambda(e_1) &= \tilde{S}^+, & \pi_\lambda(f_1) &= \tilde{S}^- \\ \pi_\lambda(k_2) &= i^{-\tilde{S}^z}, & \pi_\lambda(e_2) &= e^{-2\mu\lambda}\tilde{S}^-, & \pi_\lambda(f_2) &= e^{2\mu\lambda}\tilde{S}^+, \end{aligned} \quad (82)$$

where μ is derived from the *blob algebra* parameter q , not from $\tilde{q} = i$.

For $z \in \{q, i, r, \hat{r}\}$, consider the element of $U_z(\widehat{sl(2)})$ given by

$$\mathcal{Q}_j^z = z^{-\frac{1}{2}}k_j e_j + z^{\frac{1}{2}}k_j f_j + x_z^j k_j^2 - x_z^j I, \quad j \in \{1, 2\} \quad (83)$$

⁹Also define the co-unit ϵ and the antipode S :

$$\epsilon(e_i) = \epsilon(f_i) = 0, \quad \epsilon(k_i^{\pm 1}) = 1, \quad S(e_i) = -q^{-1}e_i, \quad S(f_i) = -qf_i, \quad S(k_i^{\pm 1}) = k_i^{\mp 1}.$$

(the scalars x_z^j will be identified later). Define $\mathcal{B} = \langle Q_1^z, Q_2^z \rangle \subset \mathcal{A}|_{q=z}$. These elements have a relatively simple expression for the iterated coproduct inherited from \mathcal{A} , i.e.

$$\Delta^n(\mathcal{Q}_i^z) = I \otimes \Delta^{(n-1)}(\mathcal{Q}_i^z) + \mathcal{Q}_i^z \otimes \Delta^{(n-1)}(k_i^2). \quad (84)$$

Note however that this is not closed on $\mathcal{B}^{\otimes n}$.

Let us now consider various actions of \mathcal{B} on $V^{\otimes 2n}$ (by restriction of the \mathcal{A} action), and hence on the solutions (41i) and (41ii) and the corresponding Hamiltonians \mathcal{H}^{Θ^I} .

Type (i) Guided by the XXZ case [47, 48] we make the following identifications for the solution (i):

$$x_q^1 = \frac{Q - Q^{-1}}{q - q^{-1}}, \quad x_q^2 = \frac{Q + Q^{-1}}{2\delta_e \sinh^2 i\mu} \cosh 2i\mu\zeta. \quad (85)$$

It can be shown by direct calculation that

$$\sigma_\lambda(\mathcal{Q}_j^q) K^{(i)}(\lambda) = K^{(i)}(\lambda) \sigma_{-\lambda}(\mathcal{Q}_j^q) \quad j \in \{1, 2\} \quad (86)$$

provided (85) holds. From (86) it can be shown that for $n = 1$

$$[\sigma_\lambda(\mathcal{Q}_1^q), \Theta^i(\mathbf{e})] = 0, \quad (87)$$

Taking into account the commutation relations (68) and (83), (87) and Proposition 7 then

$$[\sigma_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^q)), \Theta(\mathcal{U})] = [\sigma_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^q)), \Theta^i(\mathbf{e})] = 0 \quad (88)$$

(any l) so

$$[\sigma_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^q)), \mathcal{H}^{\Theta^i}] = 0. \quad (89)$$

We conclude from (89) that the presence of the boundary (41i) does not break the $U_i(sl(2))$ part of the exposed symmetry (75), and also preserves the ‘charge’ $\sigma_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^q))$ (89).

Type (ii) Considering the solution (ii) we make the identifications:

$$x_i^1 = -\frac{Q - Q^{-1}}{2i}, \quad x_i^2 = \frac{Q + Q^{-1}}{2i\delta_e \sinh i\mu} \cosh 2i\mu\zeta, \quad (90)$$

where again μ is the blob algebra parameter. Then it can be shown that

$$\pi_\lambda(\mathcal{Q}_1^i) K^{(ii)}(\lambda) = K^{(ii)}(\lambda) \pi_{-\lambda}(\mathcal{Q}_1^i) \quad (91)$$

whereas

$$\pi_\lambda(\mathcal{Q}_2^i) K^{(ii)}(\lambda) = K^{(ii)}(\lambda) \pi_{-\lambda}(\mathcal{Q}_2^i) \iff Q = e^{-\frac{i\mu}{2} - \frac{i\pi}{4}}. \quad (92)$$

From (91) it follows

$$[\pi_\lambda(\mathcal{Q}_1^i), \Theta^{ii}(\mathbf{e})] = 0. \quad (93)$$

Taking into account the commutation relations (73) and (83), (93), and Proposition 7 it can be directly shown that

$$\left[\pi_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^i)), \Theta(\mathcal{U}_i) \right] = \left[\pi_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^i)), \Theta^{ii}(\mathbf{e}) \right] = 0 \quad (94)$$

so

$$\left[\pi_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^i)), \mathcal{H}^{\Theta^{ii}} \right] = 0. \quad (95)$$

Equation (95) implies that the presence of the boundary (41ii), which breaks the $U_i(sl(2))$ (π) part of the exposed symmetry (70), preserves the ‘charge’ $\pi_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^i))$.

Symmetries in the cases (+) and (iii) are relegated to a footnote for brevity.¹⁰

6 Spectrum and Bethe ansatz equations

Our aim in what follows is to find the exact spectrum of the open asymmetric twin spin chain Hamiltonian, in types (i) and (ii), and study the role of the boundary parameter. Our objective in the remainder of the present paper is to determine the *form* of the spectrum, and to derive the Bethe ansatz equations. For numerical solutions we need the large λ asymptotics which, as already explained, will be treated elsewhere.

¹⁰ **Type (+)** The presence of boundary type (+) breaks both $U_q(sl(2))$ and $U_i(sl(2))$ symmetries. However it is clear from the form of the solution $K^{(+)}$ and from relations (74), (69), (86) and (91) (which also hold for $\Theta^i(\mathbf{e})$ (28) and $\Theta^{ii}(\mathbf{e})$ (29)) that

$$\begin{aligned} \sigma_\lambda(\mathcal{Q}_1^q) K^{(+)}(\lambda) &= K^{(+)}(\lambda) \sigma_\lambda(\mathcal{Q}_1^q) \\ \pi_\lambda(\mathcal{Q}_1^i) K^{(+)}(\lambda) &= K^{(+)}(\lambda) \pi_\lambda(\mathcal{Q}_1^i). \end{aligned} \quad (96)$$

provided that relations (85), (90) hold simultaneously. It is clear from (89), (95), (96) that

$$\left[\sigma_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^q)), \mathcal{H} \right] = 0, \quad \left[\pi_\lambda^{\otimes n}(\Delta^n(\mathcal{Q}_1^i)), \mathcal{H} \right] = 0. \quad (97)$$

Type (iii) In case (iii) both symmetries $U_q(sl(2))$, $U_i(sl(2))$ (and $U_r(sl(2))$, $U_{\hat{r}}(sl(2))$) are broken. However one can explicitly show that

$$\begin{aligned} \rho_\lambda^1(\mathcal{Q}_1^r) K^{(iii)}(\lambda) &= K^{(iii)}(\lambda) \rho_\lambda^1(\mathcal{Q}_1^r) \\ \rho_\lambda^2(\mathcal{Q}_1^{\hat{r}}) K^{(iii)}(\lambda) &= K^{(iii)}(\lambda) \rho_\lambda^2(\mathcal{Q}_1^{\hat{r}}) \end{aligned} \quad (98)$$

provided

$$x_{\hat{r}}^1 = \frac{\sqrt{iQ} - \sqrt{-iQ^{-1}}}{\hat{r} - \hat{r}^{-1}}, \quad x_r^1 = i \frac{\sqrt{iQ} + \sqrt{-iQ^{-1}}}{r - r^{-1}} \quad (99)$$

and ρ_i is given by (61). Again we obtain

$$\left[\mathcal{H}, \rho_\lambda^{1 \otimes n}(\Delta^n(\mathcal{Q}_1^r)) \right] = 0, \quad \left[\mathcal{H}, \rho_\lambda^{2 \otimes n}(\Delta^n(\mathcal{Q}_1^{\hat{r}})) \right] = 0. \quad (100)$$

It should be pointed out that we were able to derive intertwining relations (86), (91), (92) between the generators of \mathcal{B} and the K -matrices of type (i) and (ii), whereas for the other solutions we proved commutation relations only for the non-affine generators.

Reparameterizing by $Q = iq^p = ie^{i\mu p}$ and setting

$$\delta_e = -\frac{\sinh i\mu p}{\sinh(i\mu)} \quad (101)$$

we have the following formulae from (32, 33) which will be useful later

$$\kappa^i = \frac{\sinh i\mu(p-1)}{\sinh(i\mu)} \quad (102)$$

$$\kappa^{ii} = \frac{\sinh i\mu(p + \frac{\pi}{2\mu})}{\sinh(i\mu)}. \quad (103)$$

$$\kappa^{iii} = \frac{\sinh i\mu(p-1) - i}{\sinh(i\mu)} \quad (104)$$

(The parameter p in type (i) can be identified with the usual m of the parameterization $b_n(q, m)$ which appears in the literature [30]. This parameterization is natural both from the point of view of representation theory, and integrability of the corresponding spin chain. Note that p in types (i) and (ii) can be related by identifying the ratios $\frac{\delta_e}{\kappa^i}$ and $\frac{\delta_e}{\kappa^{ii}}$, but p in type (ii) does not have the same simple relation with the ‘usual’ parameterization $b_n(q, m)$. Nonetheless it is crucial for the analysis of the ‘type (ii)’ spin chain — see later.)

6.1 Reference states

Since the gemini model conserves ‘charge’ in a manner analogous to trivial boundary XXZ, the first step is to find the standard ordered basis elements of $V^{\otimes 2n}$ which are already eigenstates of the transfer matrix (45) — the pseudo-vacua. Define

$$\begin{aligned} |00\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} & |01\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} & |11\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} & |10\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \\ \\ |\omega_+\rangle &= \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{2n} = \otimes_{i=1}^n |00\rangle_i \end{aligned} \quad (105)$$

$$|\omega_2\rangle = \otimes_{i=1}^n |01\rangle_i \quad |\omega_1\rangle = \otimes_{i=1}^n |11\rangle_i \quad |\omega_3\rangle = \otimes_{i=1}^n |10\rangle_i \quad (106)$$

The pseudo-vacua $|\omega_+\rangle, |\omega_2\rangle, |\omega_1\rangle, |\omega_3\rangle$ are all eigenstates in the case $K^- \propto \mathbb{I}$.

Type (i) Consider solution (i) (41). The first reference state is the pseudo-vacuum $|\omega_+\rangle$. The pseudo-vacuum eigenvalue takes the form

$$\Lambda^0(\lambda) = f_1(\lambda)a(\lambda)^{2n} + f_2(\lambda)b(\lambda)^{2n} \quad (107)$$

where the functions $f_1(\lambda), f_2(\lambda)$ are due to the boundary (this form is derived explicitly in Appendix B: f_1, f_2 are determined explicitly by (152)–(157), (151)):

$$f_1(\lambda) = 2x(\lambda) \frac{\cosh \mu(\lambda + i) \sinh \mu(\lambda + i)}{\sinh \mu(2\lambda + i)}, \quad f_j(\lambda) = 2x_j(\lambda) \frac{\cosh \mu\lambda \sinh \mu\lambda}{\sinh \mu(2\lambda + i)} \quad j \in \{2, 3\} \quad (108)$$

where $x(\lambda)$ is given by (10) and

$$x_2(\lambda) = 2 \sinh \mu(\lambda + i - \frac{ip}{2} + i\zeta) \sinh \mu(\lambda + i - \frac{ip}{2} - i\zeta). \quad (109)$$

Note that for $\zeta \rightarrow -i\infty$ we have $\frac{1}{x}K \xrightarrow{\zeta \rightarrow -i\infty} \mathbb{I}$, $\frac{x(\lambda)-x_2(\lambda)}{x(\lambda)} \rightarrow 0$.

Type (ii) Now consider solution (ii) (41). The first reference state is $|\omega_2\rangle$. The pseudo-vacuum eigenvalue takes the form (110)

$$\Lambda^0(\lambda) = f_1(\lambda)a(\lambda)^{2n} + f_3(\lambda)b(\lambda)^{2n} \quad (110)$$

where

$$x_3(\lambda) = \frac{1}{\sinh(i\mu)} \left(\sinh(i\mu p) \cosh \mu(2\lambda + i) - \sinh i\mu(p + \frac{\pi}{2\mu}) \cosh 2\mu(\lambda + i) \right) - \cosh 2i\mu\zeta \quad (111)$$

(see Appendix B for details: applying equations (160)–(165) on (159)). The important observation here is that we were able to derive the pseudo-vacuum eigenvalue explicitly.

6.2 Duality symmetries and more reference states

The states $|\omega_2\rangle$, $|\omega_3\rangle$ (resp. $|\omega_+\rangle$, $|\omega_1\rangle$) are *not* exact eigenstates of the transfer matrix when the non-trivial right boundary (41i) (resp. (ii)) is *on* (this can be seen as a consequence of the fact that the K^- matrix (41), ((i), (ii)) breaks the $U_q(sl(2))$, $U_i(sl(2))$ part of the observed symmetry), but we could have started our construction considering $|\omega_1\rangle$ (type (i)), or $|\omega_3\rangle$ (type (ii)).

Let us briefly explain why all four reference states give the same eigenvalue when $K^- \propto \mathbb{I}$. We introduce the involutive matrices

$$W^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad W^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad W^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (112)$$

Acting by conjugation they satisfy

$$\begin{aligned} W_i^{(l)} W_j^{(l)} R_{ij}^{(l)}(\lambda; r^{-\delta_{l1}-\delta_{l3}+\delta_{l2}}, \hat{r}^{-\delta_{l1}-\delta_{l2}+\delta_{l3}}) W_i^{(l)} W_j^{(l)} &= R_{ij}^{(l)}(\lambda; r, \hat{r}), \\ W_i^{(l)} M_i^{(l)}(r^{-\delta_{l1}-\delta_{l3}+\delta_{l2}}, \hat{r}^{-\delta_{l1}-\delta_{l2}+\delta_{l3}}) W_i^{(l)} &= M_i^{(l)}(r, \hat{r}) \end{aligned} \quad (113)$$

Here $R_{ij}^{(l)}(\lambda, r, \hat{r})$ is the formal matrix given by the definition of $R_{ij}(\lambda)$ in Appendix A, but where r, \hat{r} are not merely shorthand as in Appendix A, but can be manipulated as if separate variables. Consequently

$$\mathcal{W}^{(l)} t(\lambda; r^{-\delta_{l1}-\delta_{l3}+\delta_{l2}}, \hat{r}^{-\delta_{l1}-\delta_{l2}+\delta_{l3}}) \mathcal{W}^{(l)} = t(\lambda; r, \hat{r}) \quad (114)$$

where $\mathcal{W}^{(l)} = W_1^{(l)} \dots W_n^{(l)}$ on $V_1 \otimes \dots \otimes V_n$ (recall $V_i = \mathbb{C}^4$). We call these actions dualities.¹¹ We stress that the transformations (113) leave $a(\lambda)$ and $b(\lambda)$ in the R -matrix *invariant*. Furthermore,

$$\mathcal{W}^{(l)} |\omega_+\rangle = |\omega_l\rangle, \quad (115)$$

¹¹just because they are involutions.

so

$$\mathcal{W}^{(l)} t(\lambda; r^{-\delta_{l1}-\delta_{l3}+\delta_{l2}}, \hat{r}^{-\delta_{l1}-\delta_{l2}+\delta_{l3}}) |\omega_+\rangle = t(\lambda; r, \hat{r}) |\omega_l\rangle, \quad (116)$$

but it is easy to show (taking into account Appendix B relations (151), (157)), that the pseudo-vacuum eigenvalue (107) remains invariant under the ‘functional duality’ symmetry $p \rightarrow p^{-1}$ for all $p \in \{r, \hat{r}, q\}$. This together with (116) implies that the four distinct states are degenerate for $K^- \propto \mathbb{I}$.

Only $W^{(1)}$ ‘commutes’ with the K -matrices (41i), (41ii):

$$W_i^{(1)} K_i(\lambda; -\mu) W_i^{(1)} = K_i(\lambda; \mu) \quad (117)$$

Consequently equation (116) is valid only for $l = 1$ and for the ‘duality’ $q \rightarrow q^{-1}$ in general. We conclude that the degeneracy is reduced by the non-trivial boundary. Again this corresponds to the K -matrices (41) breaking the observed symmetry of the model from $U_q(sl(2)) \otimes U_i(sl(2))$ to $U_i(sl(2))$ (resp. $U_q(sl(2))$).

Let us briefly comment on the action of the transformations (112) on the $U_q(sl(2))$, $U_i(sl(2))$ symmetries for $K^- \propto \mathbb{I}$. Let $\gamma \in \{\pi, \sigma\}$ (71), (66). Then by direct calculation

$$\begin{aligned} \mathcal{W}^{(1)} (\gamma^{\otimes n}(\Delta^n(\mathcal{K}))) \mathcal{W}^{(1)} &= \gamma^{\otimes n}(\Delta'^n(\mathcal{K})), \\ \mathcal{W}^{(1)} (\gamma^{\otimes n}(\Delta^n(\mathcal{E}))) \mathcal{W}^{(1)} &= \gamma^{\otimes n}(\Delta'^n(\mathcal{F})), \\ \mathcal{W}^{(1)} (\gamma^{\otimes n}(\Delta^n(\mathcal{F}))) \mathcal{W}^{(1)} &= \gamma^{\otimes n}(\Delta'^n(\mathcal{E})) \end{aligned} \quad (118)$$

where Δ' is derived as (53) but with $\mathcal{K} \rightarrow \mathcal{K}^{-1}$, which suggests that the $U_q(sl(2))$ $U_{\tilde{q}}(sl(2))$ symmetry is preserved under the duality transformation $\mathcal{W}^{(1)}$, as long as the co-product structure becomes Δ' . The transformations $\mathcal{W}^{(2)}$ and $\mathcal{W}^{(3)}$ modify the existing symmetry. Indeed these transformations somehow interchange the role between the $U_q(sl(2))$, $U_i(sl(2))$ symmetries. Let $\tilde{\pi}$ be defined as in (71) but with $i \rightarrow q$, then

$$\mathcal{W}^{(2)} (\sigma^{\otimes n}(\Delta^n(x))) \mathcal{W}^{(2)} = \tilde{\pi}^{\otimes n}(\Delta'^n(x)), \quad \mathcal{W}^{(3)} (\sigma^{\otimes n}(\Delta^n(x))) \mathcal{W}^{(3)} = \tilde{\pi}^{\otimes n}(\Delta^n(x)). \quad (119)$$

6.3 Derivation of the eigenvalues and the Bethe ansatz

Having obtained the pseudovacuum eigenvalues in section 6.1 we come to the derivation of the general eigenvalue. We make the following assumption for the structure of the general eigenvalue,

$$\Lambda(\lambda) = f_1(\lambda)a(\lambda)^{2n}\mathfrak{A}_1(\lambda) + f_2(\lambda)b(\lambda)^{2n}\mathfrak{A}_2(\lambda), \quad (120)$$

where $\mathfrak{A}_1(\lambda)$, $\mathfrak{A}_2(\lambda)$ are to be determined explicitly via the analytical Bethe ansatz method [52–56].

In what follows we consider the analytical rather than the algebraic Bethe ansatz formulation [2, 3] to derive the eigenvalues and the Bethe ansatz equations (i.e., basically, we observe certain constraints on the transfer matrix eigenvalues, which fix the form of the \mathfrak{A}_i ’s). We first derive the Bethe ansatz equations for trivial boundary conditions $K^\pm \propto \mathbb{I}$, then generalize to the boundary K^- given by (41).

• $K^\pm \propto \mathbb{I}$: The eigenvalues of the transfer matrix are analytic, i.e. \mathfrak{A}_i 's must have common poles. This restriction entails certain relations between \mathfrak{A}_i 's. In particular, from crossing (46)

$$\mathfrak{A}_1(-\lambda - i) = \mathfrak{A}_2(\lambda) \quad (121)$$

from the fusion relation (see Appendix C)

$$\mathfrak{A}_1(\lambda + i) \mathfrak{A}_2(\lambda) = 1 \quad (122)$$

and from the analyticity: the pole $\lambda = -\frac{i}{2}$ in the eigenvalue expression (120) must disappear, i.e.

$$\mathfrak{A}_1(-\frac{i}{2}) = \mathfrak{A}_2(-\frac{i}{2}). \quad (123)$$

From equations (121) and (122) we have

$$\mathfrak{A}_1(\lambda) \mathfrak{A}_1(-\lambda) = 1. \quad (124)$$

Since the R -matrix involves trigonometric functions only, it is implied that \mathfrak{A}_i 's can be written as products of trigonometric functions in the following way

$$\mathfrak{A}_1(\lambda) = \prod_{i=1}^{M_1} \frac{\sinh \mu(\lambda - x_i^1)}{\sinh \mu(\lambda - y_i^1)}, \quad \mathfrak{A}_2(\lambda) = \prod_{i=1}^{M_2} \frac{\sinh \mu(\lambda - x_i^2)}{\sinh \mu(\lambda - y_i^2)}. \quad (125)$$

Our task now is to determine the numbers $x_i^{1,2}, y_i^{1,2}$ by solving the aforementioned constraints. By solving these constraints we derive the form of \mathfrak{A}_i 's ($M_1 = M_2 = M$ immediately from crossing, and the fact that \mathfrak{A}_i 's must have common poles):

$$\begin{aligned} \mathfrak{A}_1(\lambda) &= \prod_{i=1}^M \frac{\sinh \mu(\lambda - \lambda_i - \frac{i}{2})}{\sinh \mu(\lambda - \lambda_i + \frac{i}{2})} \frac{\sinh \mu(\lambda + \lambda_i - \frac{i}{2})}{\sinh \mu(\lambda + \lambda_i + \frac{i}{2})}, \\ \mathfrak{A}_2(\lambda) &= \mathfrak{A}_1(-\lambda - i) \end{aligned} \quad (126)$$

Here M is still an unknown integer, and $\lambda_1, \lambda_2, \dots$ are constants to be determined. In principle, M can be determined by computing the asymptotic behavior of the transfer matrix for $t(\lambda \rightarrow \pm\infty)$. However, in our case this is rather an intriguing task because, as already mentioned, the R -matrix does not reduce to upper (lower) triangular as usual. Nevertheless, it is still possible to determine M , as we show shortly.

Let us first derive the Bethe ansatz equations which follow from the analyticity requirements for the eigenvalues (120) (the poles must cancel):

$$\left(\frac{\sinh \mu(\lambda_i + \frac{i}{2})}{\sinh \mu(\lambda_i - \frac{i}{2})} \right)^{2n} = \prod_{i \neq j=1}^M \frac{\sinh \mu(\lambda_i - \lambda_j + i)}{\sinh \mu(\lambda_i - \lambda_j - i)} \frac{\sinh \mu(\lambda_i + \lambda_j + i)}{\sinh \mu(\lambda_i + \lambda_j - i)} \quad (127)$$

Note that Bethe ansatz equations are essentially the conditions that fix the values of λ_i in expressions (120), (126), provided that M is determined. Having derived the form of the eigenvalues

and the Bethe ansatz equations we are in the position to determine M . We use Proposition 6, i.e. the fact that the open XXZ Hamiltonian and the open gemini Hamiltonian for $K^\pm \propto \mathbb{I}$ have the same spectrum (up to multiplicities) (see also [28]). Recall [12] that the spectrum and Bethe ansatz equations of the open XXZ model $K^\pm = \mathbb{I}$ are given by expressions similar to (120), (126) and (127) respectively, but with $M_{XXZ} \in \{0, \dots, \frac{N}{2}\}$. Now using the fact that the two Hamiltonians have the same spectrum we can take the derivative of (120) then derive the corresponding energies along the lines in (47) and compare them. From the comparison it immediately follows that $M = M_{XXZ} \in \{0, \dots, \frac{N}{2}\}$. Thus for the open twin spin chain with $K^\pm \propto \mathbb{I}$, M is determined.

• $K^- \not\propto \mathbb{I}$: When K^- is given by (41), (we restrict here for type (i) and (ii) solutions) the eigenvalues are modified (the functions f_1 and f_2 are changed) and therefore the Bethe ansatz equations are modified accordingly, namely the following extra factors:

Type (i)

$$\frac{\sinh \mu(\lambda_i + i\zeta + \frac{ip-i}{2})}{\sinh \mu(\lambda_i + i\zeta - \frac{ip-i}{2})} \frac{\sinh \mu(\lambda_i - i\zeta + \frac{ip-i}{2})}{\sinh \mu(\lambda_i + i\zeta - \frac{ip-i}{2})} \quad (128)$$

Type (ii)

$$\frac{\sinh(i\mu p) \cosh 2\mu\lambda - \sinh i\mu(p + \frac{\pi}{2\mu}) \cosh \mu(2\lambda - i) - \sinh i\mu \cosh 2i\mu\zeta}{\sinh(i\mu p) \cosh 2\mu\lambda - \sinh i\mu(p + \frac{\pi}{2\mu}) \cosh \mu(2\lambda + i) - \sinh i\mu \cosh 2i\mu\zeta} \quad (129)$$

multiply the LHS of the Bethe ansatz equation (127) for solutions (i) and (ii) respectively.

Note that in case the right boundary is non-trivial, as in solutions (41) (i) (ii), we are restricted to the eigenvalues entailed from the subset of pseudo-vacua $\{|\omega_+\rangle, |\omega_1\rangle\}$, or $\{|\omega_2\rangle, |\omega_3\rangle\}$ respectively (since the other ‘reference’ states are no longer exact eigenstates). Of course the total number of eigenvalues is unchanged, but the non-trivial boundary breaks the symmetry of the trivial boundary model, reducing the initial degeneracy. Thus, in addition to the eigenvalues computed starting from $|\omega_+\rangle$, $(|\omega_1\rangle)$ or $|\omega_2\rangle$, $(|\omega_3\rangle)$ as pseudo-vacua, there may exist further eigenvalues entailed from more complicated reference eigenstates. One can address the computation of these eigenvalues following the methods described in [29, 57].

Finally the derivation of M in case $K^- \neq \mathbb{I}$ is a more complicated problem than $K^- = \mathbb{I}$. However, by Proposition 6 the spectrum of the open gemini spin chain is the same as the spectrum of the open XXZ spin chain with a non-diagonal boundary (see also [28]). So again one has to compare the Hamiltonian eigenvalues by taking the derivative of the transfer matrix eigenvalues (47). The remaining problem is, in effect, to derive the eigenvalues of the XXZ chain with one non-diagonal boundary (in the ‘homogeneous gradation’). We will treat this in a separate work.

7 Problems and discussion

7.1 Bulk case $n = 2$: Eigenvalues

There are limits to the power of proposition 6. For example the bulk periodic case, i.e. with transfer matrix [2, 3]

$$t(\lambda) = \text{tr}_0 T_0(\lambda) \quad (130)$$

cannot be expressed directly in the algebraic form (see [58]). Thus the proposition does not force the bulk gemini chain to have the same spectrum as the corresponding well known XXZ chain. We shall show that it does not, by determining the eigenvalues of the 16×16 periodic $n = 2$ gemini transfer matrix:

$$\begin{pmatrix} \begin{array}{ccc|ccc|ccc|ccc} a^2+b^2 & & & & & & & & & & & \\ & 0 & & & & & & & & & & \\ & & 0 & & & & & & & & & \\ & & & 2ab & & & & & & & & \\ \hline & a^2+b^2 & & & & & & & & & & \\ & & -(s+\frac{1}{s})ab & & & & & & & & & \\ & & & 2ab & & & & & & & & \\ & & & & 0 & & & & & & & \\ \hline & a^2+b^2 & & & & & & & & & & \\ & & -(r+\frac{1}{r})ab & & & & & & & & & \\ & & & (a-qb)(a-\frac{b}{q}) & & & & & & & & \\ & & & & 2ab & & & & & & & \\ & & & & & a^2+b^2 & & & & & & \\ & & & & & & 0 & & & & & \\ \hline & a^2+b^2 & & & & & & & & & & \\ & & -(r+\frac{1}{r})ab & & & & & & & & & \\ & & & a^2+b^2 & & & & & & & & \\ & & & & & -(s+\frac{1}{s})ab & & & & & & \\ & & & & & & 2ab & & & & & \\ & & & & & & & 0 & & & & \\ & & & & & & & & 0 & & & \\ & & & & & & & & & a^2+b^2 & & \end{array} \end{pmatrix}$$

where

$$a = a(\lambda) = \sinh \mu(\lambda + i), \quad b = b(\lambda) = \sinh \mu\lambda. \quad (131)$$

As before there are eigenstates of the form $|\omega_+\rangle$, and also of the form

$$|\omega_{5,6}\rangle = |00\rangle \otimes |01\rangle \pm |01\rangle \otimes |00\rangle \quad (132)$$

The corresponding eigenvalues are $a^2(\lambda) + b^2(\lambda)$ (8-fold degenerate), and $-(a^2(\lambda) + b^2(\lambda))$, (4-fold degenerate). The last four eigenstates are derived by diagonalizing the 4×4 block in t . Consider possible eigenstates of the form

$$|\omega_j\rangle \equiv \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix}. \quad (133)$$

Here x, y, z must satisfy the following equations

$$\left((r + r^{-1})(x - yz) + (\hat{r} + \hat{r}^{-1})(y - xz) \right) a(\lambda)b(\lambda) = \left(a^2(\lambda) + b^2(\lambda) \right) (1 - z^2) \quad (134)$$

$$\left((r + r^{-1})(1 - y^2) + (\hat{r} + \hat{r}^{-1})(z - xy) + (q + q^{-1})x \right) a(\lambda)b(\lambda) = \left(a^2(\lambda) + b^2(\lambda) \right) (x - zy) \quad (135)$$

$$\left((r + r^{-1})(z - yx) + (\hat{r} + \hat{r}^{-1})(1 - x^2) + (q + q^{-1})y\right)a(\lambda)b(\lambda) = \left(a^2(\lambda) + b^2(\lambda)\right)(y - zx) \quad (136)$$

This yields the following eigenvalues: For $x = y, z = 1$ we have:

$$\epsilon_1^+(\lambda) = 4a(\lambda)b(\lambda) + \left(a(\lambda) - qb(\lambda)\right)\left(a(\lambda) - q^{-1}b(\lambda)\right), \quad \epsilon_2^+(\lambda) = a^2(\lambda) + b^2(\lambda) \quad (137)$$

For $x = -y, z = -1$:

$$\epsilon_1^-(\lambda) = 4a(\lambda)b(\lambda) - \left(a(\lambda) - qb(\lambda)\right)\left(a(\lambda) - q^{-1}b(\lambda)\right), \quad \epsilon_2^-(\lambda) = -a^2(\lambda) - b^2(\lambda). \quad (138)$$

Compare these eigenvalues for the gemini periodic chain with those for periodic XXZ for $n = 2$:

$$\epsilon(\lambda) = a^2(\lambda) + b^2(\lambda) \text{ (2-fold degenerate), } \epsilon_{\pm}(\lambda) = 2a(\lambda)b(\lambda) \pm \left(a(\lambda) - qb(\lambda)\right)\left(a(\lambda) - q^{-1}b(\lambda)\right) \quad (139)$$

Ignoring the irrelevant multiplicities, it is interesting to note that the spectra do not coincide. There is often a close relation between the bulk and open transfer matrix eigenvalues: the open transfer matrix eigenvalues are ‘doubled’ compared to the bulk ones, i.e. if the bulk eigenvalues have the form

$$\Lambda_{bulk}(\lambda) = a(\lambda)^N \mathfrak{A}_1(\lambda) + b(\lambda)^N \mathfrak{A}_2(\lambda) \quad (140)$$

then

$$\Lambda_{open}(\lambda) = f_1(\lambda) a(\lambda)^{2N} \tilde{\mathfrak{A}}_1(\lambda) + f_2(\lambda) b(\lambda)^{2N} \tilde{\mathfrak{A}}_2(\lambda) \quad (141)$$

where the functions f_i are due to the boundaries. As shown in the previous section the spectrum for open gemini and open XXZ spin chain are the same. Hence, considering (140), (141), one might have expected a similar statement for the bulk case as well.

7.2 The XXZ versus the gemini R -matrix

Recall that the XXZ R -matrix is given on $\mathbb{C}^2 \otimes \mathbb{C}^2$ by

$$R(\lambda) = \mathcal{P} \left(\sinh \mu(\lambda + i) + \sinh \mu\lambda U(q) \right) \quad (142)$$

Let

$$\Pi : \mathcal{X}_1 \otimes \mathcal{X}_2 \rightarrow \mathcal{X}_2 \otimes \mathcal{X}_1, \quad (143)$$

and also define

$$\Delta'(x) = \Pi \circ \Delta(x), \quad x \in \mathcal{A}. \quad (144)$$

Then the commutation (59) can be restated in the well known form

$$R_{12}(\lambda) \rho^{\otimes 2}(\Delta(x)) = \rho^{\otimes 2}(\Delta'(x)) R_{12}(\lambda), \quad \forall x \in U_q(sl(2)). \quad (145)$$

where recall ρ is the 2d representation of $U_q(sl(2))$. We also have the following intertwining relations between the representations $(\rho_\lambda \otimes \rho_0) \Delta(x)$ and $(\rho_\lambda \otimes \rho_0) \Delta'(x)$:

$$R_{12}(\lambda) (\rho_\lambda \otimes \rho_0) \Delta(x) = (\rho_\lambda \otimes \rho_0) \Delta'(x) R_{12}(\lambda) \quad \forall x \in \mathcal{A}. \quad (146)$$

(It is in this sense that the R -matrix is associated [51, 59] with $U_q(\widehat{sl(2)})$.)

Relations (145), (146) were first introduced in [51, 59], establishing the quantum group approach in obtaining solutions of the Yang-Baxter equation (1). They also play a crucial role in the study of the underlying symmetries in 2D relativistic integrable field theories [60], and they have been extensively used for computing the corresponding exact S -matrices (see e.g. [60]). Relation (146) takes a simple form for $\lambda \rightarrow \pm\infty$, namely

$$(\sigma^\pm \otimes q^{\frac{1}{2}\sigma^z}) R_\pm = R_\pm (\sigma^\pm \otimes q^{-\frac{1}{2}\sigma^z}), \quad (q^{-\frac{1}{2}\sigma^z} \otimes \sigma^\mp) R_\pm = R_\pm (q^{\frac{1}{2}\sigma^z} \otimes \sigma^\mp), \quad (147)$$

where $R_\pm = R_{12}(\pm\infty)$. Moreover, the XXZ R -matrix reduces to upper (lower) triangular matrix as $\lambda \rightarrow \pm\infty$, which makes the study of the asymptotic behavior of the transfer matrix (45) and its symmetry relatively easy.

One can show that relations of the type (145) are also valid for the gemini R -matrix (38) and the representations ρ_i , π and σ defined in (61), (71), (66). Indeed let $h \in \{\pi, \sigma, \rho_i\}$ then it can be shown by straightforward computation that

$$h^{\otimes 2}(\Delta'(x)) R_{12}(\lambda) = R_{12}(\lambda) h^{\otimes 2}(\Delta(x)), \quad \forall x \in \mathcal{G} \quad (148)$$

where R is the gemini matrix. In fact (148) is just an ‘unchecked’ restatement of the various commutations.

Straightforward generalizations of the evaluation representation (along the lines in (79)) are π_λ , σ_λ , ρ_λ^1 , ρ_λ^2 . The derivation of a result analogous to (146) remains an open problem (the obvious constructions based on the representations above do not work). This means that so far we have not been able to see the relation of our R -matrix with a corresponding quantum affine algebra. It is this, together with the fact that the R -matrix does *not* reduce to an upper (lower) triangular matrix for $\lambda \rightarrow \pm\infty$, which makes the study of the asymptotic behavior of the transfer matrix (45) and its symmetry such an intriguing task.

Although the study of the asymptotic behavior of the R -matrix is complicated for generic values of q , one can exploit relations (145) and study the symmetry of the open transfer matrix along the lines described in [47, 48].

Martin and Saleur’s original tensor space b_n representation [28] has also been discussed recently in [61].

Acknowledgements: A.D. is supported by the European Network “EUCLID. Integrable models and applications: from strings to condensed matter”, contract number HPRN-CT-2002-00325.

Appendix A The explicit R -matrix

Firstly we write $\Theta(\mathcal{U}_1)$ as a 16×16 matrix acting *not* on $V_{2-} \otimes V_{1-} \otimes V_{1+} \otimes V_{2+}$ as in (27), but on $(V_{1+} \otimes V_{1-}) \otimes (V_{2+} \otimes V_{2-})$

$$\Theta(\mathcal{U}_1) = \begin{pmatrix} 0 & & & & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & & & & \\ & & 0 & & & & & & & & & & & & & \\ & & & -i & & -r^{-1} & & -\hat{r} & & 1 & & & & & & \\ & & & & 0 & & & & & & & & & & & \\ & & & & & 0 & & & & & & & & & & \\ & & & -r^{-1} & & -q^{-1} & & 1 & & -\hat{r}^{-1} & & & & & & \\ & & & & & & 0 & & & & & & & & & \\ & & & & & & & 0 & & & & & & & & \\ & & & -\hat{r} & & 1 & & -q & & -r & & & & & & \\ & & & & & & & & 0 & & & & & & & \\ & & & & & & & & & 0 & & & & & & \\ & & & 1 & & -\hat{r}^{-1} & & -r & & i & & & & & & \\ & & & & & & & & & & 0 & & & & & \\ & & & & & & & & & & & 0 & & & & \\ & & & & & & & & & & & & 0 & & & \end{pmatrix}$$

$$a1 + b\Theta(\mathcal{U}_1) = \begin{pmatrix} a & & & & & & & & & & & & & & & \\ & a & & & & & & & & & & & & & & \\ & & a & & & & & & & & & & & & & \\ & & & a - ib & & -r^{-1}b & & -\hat{r}b & & b & & & & & & \\ & & & & a & & & & & & & & & & & \\ & & & -r^{-1}b & & a - q^{-1}b & & b & & -\hat{r}^{-1}b & & & & & & \\ & & & & & & a & & & & & & & & & \\ & & & -\hat{r}b & & b & & a - qb & & -rb & & & & & & \\ & & & & & & & & a & & & & & & & \\ & & & b & & -\hat{r}^{-1}b & & -rb & & a + ib & & & & & & \\ & & & & & & & & & & a & & & & & \\ & & & & & & & & & & & a & & & & \\ & & & & & & & & & & & & a & & & \end{pmatrix}$$

$$\mathcal{P}(a1 + b\Theta(\mathcal{U}_1)) = \begin{pmatrix} a & & & \\ & 0 & & \\ & & 0 & \\ & & & b \\ \hline a & & & \\ & 0 & & \\ & & a & \\ & & & b \\ & & & & 0 \\ \hline a & & & \\ & -r^{-1}b & & \\ & & a - q^{-1}b & \\ & & & 0 \\ \hline a - ib & & & \\ & -r^{-1}b & & \\ & & a & \\ & & & a \\ & & & & 0 \\ & & & & & 0 \\ & & & & & & a \end{pmatrix}$$

The basis is $\{ijkl \mid i, j, k, l \in \{1, 2\}\}$, and we will assign the obvious standard order. Thus $\mathcal{P}ijkl = kl ij$. Also

$$M = \text{diagonal}(i, q^{-1}, q, -i).$$

It is convenient for what follows to write the above asymmetric twin R -matrix (cf. (38)) as a 4×4 matrix with 4×4 matrix entries; and give names to the corresponding blocks of the monodromy matrix T (44):

$$R(\lambda) = \begin{pmatrix} A(\lambda) & B_1(\lambda) & B_2(\lambda) & B(\lambda) \\ C_1(\lambda) & A_1(\lambda) & B_5(\lambda) & B_3(\lambda) \\ C_2(\lambda) & C_5(\lambda) & A_2(\lambda) & B_4(\lambda) \\ C(\lambda) & C_3(\lambda) & C_4(\lambda) & D(\lambda) \end{pmatrix}, \quad T_0(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}_1(\lambda) & \mathcal{B}_2(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}_1(\lambda) & \mathcal{A}_1(\lambda) & \mathcal{B}_5(\lambda) & \mathcal{B}_3(\lambda) \\ \mathcal{C}_2(\lambda) & \mathcal{C}_5(\lambda) & \mathcal{A}_2(\lambda) & \mathcal{B}_4(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{C}_3(\lambda) & \mathcal{C}_4(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} \quad (149)$$

Appendix B Pseudo-vacuum eigenvalues

Here we derive the action of the monodromy matrices and the transfer matrix on the pseudo-vacua $|\omega_+\rangle$ and $|\omega_2\rangle$.

(i) The action on $|\omega_+\rangle$: From the action of the R matrix on the pseudo-vacuum $|+\rangle$ we get: $A_i, C_i, B_5|+\rangle = 0$, i.e. $|\omega_+\rangle$ is annihilated by the operators $\mathcal{A}_i, \mathcal{C}_i, \mathcal{B}_5$. Therefore,

$$\hat{T}_0(\lambda) |\omega_+\rangle = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}'_1(\lambda) & \mathcal{B}'_2(\lambda) & \mathcal{B}'(\lambda) \\ 0 & 0 & 0 & \mathcal{B}'_3(\lambda) \\ 0 & 0 & 0 & \mathcal{B}'_4(\lambda) \\ 0 & 0 & 0 & \mathcal{D}(\lambda) \end{pmatrix} |\omega_+\rangle. \quad (150)$$

Then the pseudo-vacuum eigenvalue will be

$$\begin{aligned} \Lambda^0(\lambda) &= \langle \omega_+ | \left(-r^{-1}\hat{r}x(\lambda; m)\mathcal{A}^2 - r\hat{r}^{-1}x(\lambda; m)\mathcal{D}^2 - r\hat{r}^{-1}x(\lambda; m)\mathcal{C}\mathcal{B}' - r^{-1}\hat{r}^{-1}x(\lambda; m)\mathcal{C}_1\mathcal{B}'_1 \right. \\ &\quad \left. - r\hat{r}x(\lambda; m)\mathcal{C}_2\mathcal{B}'_2 - r\hat{r}^{-1}w^-(\lambda)\mathcal{C}_3\mathcal{B}'_3 - r\hat{r}^{-1}w^+(\lambda)\mathcal{C}_4\mathcal{B}'_4 \right) | \omega_+ \rangle. \end{aligned} \quad (151)$$

The actions of \mathcal{A} , \mathcal{D} , \mathcal{B}_i , \mathcal{C}_i , \mathcal{B} , \mathcal{C} on $|\omega_+\rangle$ are

$$\mathcal{A}(\lambda)|\omega_+\rangle = \prod_{l=1}^n A^{\bar{l}}|\omega_+\rangle, \quad \mathcal{D}(\lambda)|\omega_+\rangle = \prod_{l=1}^n D^{\bar{l}}|\omega_+\rangle \quad (152)$$

where $A^{\bar{l}} = 1 \otimes 1 \dots \otimes A(\lambda) \otimes \dots$ etc.,

$$\begin{aligned} \mathcal{C}_{1,2}(\lambda)|\omega_+\rangle &= \prod_{l=1}^{n-1} A^{\bar{l}} C_{1,2}^{\bar{n}}|\omega_+\rangle, & \mathcal{B}_{1,2}(\lambda)|\omega_+\rangle &= \prod_{l=1}^{n-1} A^{\bar{l}} B_{1,2}^{\bar{n}}|\omega_+\rangle \\ \mathcal{C}_{3,4}(\lambda)|\omega_+\rangle &= \prod_{l=2}^n D^{\bar{l}} C_{3,4}^{\bar{1}}|\omega_+\rangle, & \mathcal{B}_{3,4}(\lambda)|\omega_+\rangle &= \prod_{l=2}^n D^{\bar{l}} B_{3,4}^{\bar{1}}|\omega_+\rangle \end{aligned} \quad (153)$$

and

$$\begin{aligned} \mathcal{C}(\lambda)|\omega_+\rangle &= \left(\sum_{l=1}^n D^{\bar{n}} \dots \widetilde{D^{\bar{l+1}}} C^{\bar{l}} \widetilde{A^{\bar{l-1}}} \dots A^{\bar{1}} + \sum_{l=1}^{n-1} D^{\bar{n}} \dots \widetilde{D^{\bar{l+2}}} C_4^{\bar{l+1}} \widetilde{C_2^{\bar{l}}} \widetilde{A^{\bar{l-1}}} \dots A^{\bar{1}} \right. \\ &\quad \left. + \sum_{l=1}^{n-1} D^{\bar{n}} \dots \widetilde{D^{\bar{l+2}}} C_3^{\bar{l+1}} \widetilde{C_1^{\bar{l}}} \widetilde{A^{\bar{l-1}}} \dots A^{\bar{1}} \right) |\omega_+\rangle \end{aligned} \quad (154)$$

$$\begin{aligned} \mathcal{B}(\lambda)|\omega_+\rangle &= \left(\sum_{l=1}^n D^{\bar{n}} \dots \widetilde{D^{\bar{l+1}}} B^{\bar{l}} \widetilde{A^{\bar{l-1}}} \dots A^{\bar{1}} + \sum_{l=1}^{n-1} D^{\bar{n}} \dots \widetilde{D^{\bar{l+2}}} B_4^{\bar{l+1}} \widetilde{B_2^{\bar{l}}} \widetilde{A^{\bar{l-1}}} \dots A^{\bar{1}} \right. \\ &\quad \left. + \sum_{l=1}^{n-1} D^{\bar{n}} \dots \widetilde{D^{\bar{l+2}}} B_3^{\bar{l+1}} \widetilde{B_1^{\bar{l}}} \widetilde{A^{\bar{l-1}}} \dots A^{\bar{1}} \right) |\omega_+\rangle \end{aligned} \quad (155)$$

The primed operators are similar to the operators derived in the latter equations but with the parameters r , \hat{r} , $q \rightarrow r^{-1}$, \hat{r}^{-1} , q^{-1} . It is also useful to derive the local action of the following operators on the $|+\rangle$ state:

$$\begin{aligned} A^2|+\rangle &= a^2(\lambda)|+\rangle, & D^2|+\rangle &= b^2(\lambda)|+\rangle, \\ C_1 B'_1|+\rangle &= a^2(\lambda)|+\rangle, & C_2 B'_2|+\rangle &= a^2(\lambda)|+\rangle, \\ C B'|+\rangle &= \left(a(\lambda) + r^{-1} \hat{r} b(\lambda) \right)^2 |+\rangle, \\ C_3 B'_3|+\rangle &= r^{-2} b^2(\lambda)|+\rangle, & C_4 B'_4|+\rangle &= \hat{r}^2 b^2(\lambda)|+\rangle. \end{aligned} \quad (156)$$

(the primed operators are the usual operators given in Appendix A with $r, \hat{r}, q \rightarrow r^{-1}, \hat{r}^{-1}, q^{-1}$). Taking into account equations (152)–(156) we conclude that

$$\begin{aligned} \mathcal{A}^2|\omega_+\rangle &= a^{2n}|\omega_+\rangle, & \mathcal{D}^2|\omega_+\rangle &= b^{2n}|\omega_+\rangle \\ \mathcal{C}_1 \mathcal{B}'_1|\omega_+\rangle &= \mathcal{C}_2 \mathcal{B}'_2|\omega_+\rangle = a^{2n}|\omega_+\rangle \\ \mathcal{C}_3 \mathcal{B}'_3|\omega_+\rangle &= r^{-2} b^{2n}|\omega_+\rangle, & \mathcal{C}_4 \mathcal{B}'_4|\omega_+\rangle &= \hat{r}^2 b^{2n}|\omega_+\rangle \\ \mathcal{C} \mathcal{B}'|\omega_+\rangle &= \left((a + r^{-1} \hat{r} b)^2 \frac{a^{2n} - b^{2n}}{a^2 - b^2} + (r^{-2} + \hat{r}^2) \frac{b^2 a^{2n} - a^2 b^{2n}}{a^2 - b^2} \right) |\omega_+\rangle. \end{aligned} \quad (157)$$

(ii) **The action on $|\omega_2\rangle$:** From the action of the R matrix on the second pseudo-vacuum $|2\rangle$ we get: $A, D, C_2, C_3, B_1, B_4, B, C|2\rangle = 0$. Therefore,

$$\hat{T}_0(\lambda)|\omega_2\rangle = \begin{pmatrix} 0 & 0 & \mathcal{B}'_2(\lambda) & 0 \\ \mathcal{C}'_1(\lambda) & \mathcal{A}_1(\lambda) & \mathcal{B}'_5(\lambda) & \mathcal{B}'_3(\lambda) \\ 0 & 0 & \mathcal{A}_2(\lambda) & 0 \\ 0 & 0 & \mathcal{C}'_4(\lambda) & 0 \end{pmatrix} |\omega_2\rangle. \quad (158)$$

Then the pseudo-vacuum eigenvalue will be

$$\begin{aligned} \Lambda^0(\lambda) &= \langle \omega_2 | \left(q^{-1}x(\lambda; m)\mathcal{A}_1^2 + qx(\lambda; m)\mathcal{A}_2^2 + ix(\lambda; m)\mathcal{B}_1\mathcal{C}'_1 + qx(\lambda; m)\mathcal{C}_5\mathcal{B}'_5 \right. \\ &\quad \left. - ix(\lambda; m)\mathcal{C}_3\mathcal{B}'_3 + q\tilde{w}^-(\lambda)\mathcal{C}_2\mathcal{B}'_2 + q\tilde{w}^+(\lambda)\mathcal{B}_4\mathcal{C}'_4 \right) | \omega_2 \rangle. \end{aligned} \quad (159)$$

The action of $\mathcal{A}, \mathcal{D}, \mathcal{B}_i, \mathcal{C}_i, \mathcal{B}, \mathcal{C}$ on $|\omega_2\rangle$ are given in below.

$$\mathcal{A}_i(\lambda)|\omega_2\rangle = \prod_{l=1}^n A_i^{\tilde{l}}|\omega_2\rangle, \quad (160)$$

$$\begin{aligned} \mathcal{C}_1(\lambda)|\omega_2\rangle &= \prod_{l=1}^{n-1} A_1^{\tilde{l}} C_1^{\tilde{n}}|\omega_2\rangle, & \mathcal{B}_1(\lambda)|\omega_2\rangle &= \prod_{l=1}^{n-1} A_1^{\tilde{l}} B_{1,2}^{\tilde{n}}|\omega_2\rangle \\ \mathcal{B}_3(\lambda) &= \prod_{l=1}^{n-1} A_1^{\tilde{l}} B_3^{\tilde{n}}, & \mathcal{C}_3(\lambda) &= \prod_{l=1}^{n-1} A_1^{\tilde{l}} C_3^{\tilde{n}} \\ \mathcal{C}_4(\lambda) &= \prod_{l=2}^n A_2^{\tilde{l}} C_4^{\tilde{1}}, & \mathcal{B}_4(\lambda) &= \prod_{l=2}^n A_2^{\tilde{l}} B_4^{\tilde{1}} \\ \mathcal{B}_2(\lambda) &= \prod_{l=2}^n A_2^{\tilde{l}} B_2^{\tilde{1}}, & \mathcal{C}_2(\lambda) &= \prod_{l=2}^n A_2^{\tilde{l}} C_2^{\tilde{1}} \end{aligned} \quad (161)$$

and

$$\begin{aligned} \mathcal{C}_5(\lambda) &= \sum_{l=1}^n A_2^{\tilde{n}} \dots \widetilde{A_2^{l+1}} \widetilde{C_5^{\tilde{l}}} \widetilde{A_1^{l-1}} \dots A_1^{\tilde{1}} + \sum_{l=1}^{n-1} A_2^{\tilde{n}} \dots \widetilde{A_2^{l+2}} \widetilde{C_2^{l+1}} \widetilde{B_1^{\tilde{l}}} \widetilde{A_1^{l-1}} \dots A_1^{\tilde{1}} \\ &+ \sum_{l=1}^{n-1} A_2^{\tilde{n}} \dots \widetilde{A_2^{l+2}} \widetilde{B_4^{l+1}} \widetilde{C_3^{\tilde{l}}} \widetilde{A_1^{l-1}} \dots A_1^{\tilde{1}} \end{aligned} \quad (162)$$

$$\begin{aligned} \mathcal{B}_5(\lambda) &= \sum_{l=1}^n A_2^{\tilde{n}} \dots \widetilde{A_2^{l+1}} \widetilde{B_5^{\tilde{l}}} \widetilde{A_1^{l-1}} \dots A_1^{\tilde{1}} + \sum_{l=1}^{n-1} A_2^{\tilde{n}} \dots \widetilde{A_2^{l+2}} \widetilde{B_2^{l+1}} \widetilde{C_1^{\tilde{l}}} \widetilde{A_1^{l-1}} \dots A_1^{\tilde{1}} \\ &+ \sum_{l=1}^{n-1} A_2^{\tilde{n}} \dots \widetilde{A_2^{l+2}} \widetilde{C_4^{l+1}} \widetilde{B_3^{\tilde{l}}} \widetilde{A_1^{l-1}} \dots A_1^{\tilde{1}} \end{aligned} \quad (163)$$

It is also useful to derive the local action of the following operators on the $|\omega^{(2)}\rangle$ state:

$$\begin{aligned}
A_1^2|\omega_2\rangle &= a^2(\lambda)|\omega_2\rangle, & A_2^2|\omega_2\rangle &= b^2(\lambda)|\omega_2\rangle, \\
C_3B_3'|\omega_2\rangle &= a^2(\lambda)|\omega_2\rangle, & B_1C_1'|\omega_2\rangle &= a^2(\lambda)|\omega_2\rangle, \\
C_5B_5'|\omega_2\rangle &= \left(a(\lambda) + r^{-1}\hat{r}^{-1}b(\lambda)\right)^2|\omega_2\rangle, \\
C_2B_2'|\omega_2\rangle &= r^{-2}b^2(\lambda)|\omega_2\rangle, & B_4C_4'|\omega_2\rangle &= \hat{r}^{-2}b^2(\lambda)|\omega_2\rangle,
\end{aligned} \tag{164}$$

and consequently

$$\begin{aligned}
\mathcal{A}_1^2|\omega_2\rangle &= a^{2n}(\lambda)|\omega_2\rangle, & \mathcal{A}_2^2|\omega_2\rangle &= b^{2nn}(\lambda)|\omega_2\rangle, \\
\mathcal{C}_3\mathcal{B}_3'|\omega_2\rangle &= a^2(\lambda)|\omega_2\rangle, & \mathcal{B}_1\mathcal{C}_1'|\omega_2\rangle &= a^{2n}(\lambda)|\omega_2\rangle, \\
\mathcal{C}_5\mathcal{B}_5'|\omega_2\rangle &= \left((a(\lambda) + r^{-1}\hat{r}^{-1}b(\lambda))^2 \frac{a^{2n} - b^{2n}}{a^2 - b^2} + (r^{-2} + \hat{r}^{-2}) \frac{b^2 a^{2n} - a^2 b^{2n}}{a^2 - b^2}\right)|\omega_2\rangle, \\
\mathcal{C}_2\mathcal{B}_2'|\omega_2\rangle &= r^{-2}b^{2n}(\lambda)|\omega_2\rangle, & \mathcal{B}_4\mathcal{C}_4'|\omega_2\rangle &= \hat{r}^{-2}b^{2n}(\lambda)|\omega_2\rangle
\end{aligned} \tag{165}$$

(note that $r^{-2} + \hat{r}^{-2} = 0$).

Appendix C Fusion procedure

Relation (122) may be derived using the fusion formalism [35, 39, 55]. We refer the reader to [35, 39, 55] for the general procedure. Here we note (concentrating on $K^+ = \mathbb{I}$) that this carries over to our open gemini spin chain.

The fused transfer matrix [35, (4.17)] is written as

$$\tilde{t}(\lambda) = \zeta(2\lambda + 2i) \, t(\lambda) \, t(\lambda + i) - \delta[T(\lambda)] \, \delta[\hat{T}(\lambda)] \, \Delta[K^-(\lambda)] \, \Delta[K^+(\lambda)] \tag{166}$$

where we define the quantum determinants

$$\begin{aligned}
\delta[T(\lambda)] &= \text{tr}_{12} \left\{ Q_{12} \, T_1(\lambda) \, T_2(\lambda + i) \right\} \\
\delta[\hat{T}(\lambda)] &= \text{tr}_{12} \left\{ Q_{12} \, \hat{T}_2(\lambda) \, \hat{T}_1(\lambda + i) \right\} \\
\Delta[K^-(\lambda)] &= \text{tr}_{12} \left\{ Q_{21} \, V_1 \, V_2 \, K_1^-(\lambda) \, R_{21}(2\lambda + i) \, K_2^-(\lambda + i) \right\} \\
\Delta[K^+(\lambda)] &= \text{tr}_{12} \left\{ Q_{12} \, V_1 \, V_2 \, M_2^{-1} \, R_{12}(-2\lambda - 3i) \, M_2 \right\}
\end{aligned} \tag{167}$$

(here for simplicity we dropped the tilde from the indices i.e. $\tilde{i} \rightarrow i$). The key feature is the quantity Q_{12} :

$$Q_{12} = -\frac{1}{2 \cosh i\mu} \Theta(\mathcal{U}_1). \tag{168}$$

This is a 16×16 matrix but, as in the XXZ case, it is a projector onto an one-dimensional space (see Appendix A or [26]) After some cumbersome algebra we conclude that

$$\delta[T(\lambda)] = \delta[\hat{T}(\lambda)] = \zeta(\lambda + i)^n, \quad \Delta[K^+(\lambda)] = g(-2\lambda - 3i), \tag{169}$$

and for $K^- = \mathbb{I}$, $\Delta[K^-(\lambda)] = g(2\lambda + i)$ where

$$\zeta(\lambda) = \sinh \mu(\lambda + i) \sinh \mu(-\lambda + i), \quad g(\lambda) = \sinh \mu(-\lambda + i). \quad (170)$$

(NB, for K^- non-trivial (as in (41i), (41ii)) the function g is apparently modified.)

Having calculated the quantum determinants the following expression for the fused transfer matrix holds (case $K^\pm = \mathbb{I}$)

$$\tilde{t}(\lambda) = \zeta(2\lambda + 2i) t(\lambda) t(\lambda + i) - \zeta(\lambda + i)^{2n} g(2\lambda + i) g(-2\lambda - 3i) \quad (171)$$

We may then return to the general formalism of [35, 39, 55] to obtain (122).

Appendix D The Hamiltonian

It is interesting to write the Hamiltonian (50), (49), in terms of Pauli matrices. First for $K^- = \mathbb{I}$,

$$\begin{aligned} \mathcal{H} = & -\frac{1}{8} \sum_{i=1}^{n-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh i\mu_r \sigma_i^z \sigma_{i+1}^z - \cosh i\mu_r I - \sinh i\mu_r (\sigma_{i+1}^z - \sigma_i^z) \right) \\ & \times \left(\sigma_{i'}^x \sigma_{(i+1)'}^x + \sigma_{i'}^y \sigma_{(i+1)'}^y + \cosh i\mu_{\hat{r}} \sigma_{i'}^z \sigma_{(i+1)'}^z - \cosh i\mu_{\hat{r}} I + \sinh i\mu_{\hat{r}} (\sigma_{(i+1)'}^z - \sigma_{i'}^z) \right) \\ & - \frac{(n-1) \cosh i\mu}{2} I + \frac{1}{4 \cosh i\mu} I_n \otimes I_{n'} \end{aligned} \quad (172)$$

For $K^- \neq \mathbb{I}$ we have to take into account the extra boundary term in the Hamiltonian

$$\delta \mathcal{H} = -\frac{\sinh(i\mu)}{4\mu x(\lambda)} \frac{d}{d\lambda} K^-(\lambda)|_{\lambda=0}. \quad (173)$$

Let us also write the Hamiltonian of the open spin chain (50), (49) for K^- given e.g. by (41), in terms of the generators we introduced in section 4.4. Define

$$S^x = \frac{1}{2}(S^+ + iS^-), \quad S^y = \frac{1}{2}(S^+ - iS^-) \quad (174)$$

and similarly for $\tilde{S}^{x,y}$. Then

$$\begin{aligned} \mathcal{H} = & -\frac{1}{4} \sum_{l=1}^{n-1} (S_l^x \widetilde{S_{l+1}^x} + S_l^y \widetilde{S_{l+1}^y} + 4 \cosh i\mu S_l^z \widetilde{S_{l+1}^z} + \cosh i\mu \mathbb{I}^{(1)}) + \frac{\sinh i\mu}{2} (\widetilde{\mathbb{I}_{n-1}^{(1)}} S_n^z - S_1^z \mathbb{I}_2^{(1)}) \\ & - \frac{1}{4} \sum_{l=1}^{n-1} (\tilde{S}_l^x \widetilde{\tilde{S}_{l+1}^x} + \tilde{S}_l^y \widetilde{\tilde{S}_{l+1}^y}) + \frac{i}{2} (\widetilde{\mathbb{I}_{n-1}^{(2)}} \tilde{S}_n^z - \tilde{S}_1^z \mathbb{I}_2^{(2)}) \\ & - \frac{1}{2} \sum_{l=1}^{n-1} \left(r^{-1}(e_{12})_{\bar{l}} (e_{43})_{\widetilde{l+1}} + r^{-1}(e_{21})_{\bar{l}} (e_{34})_{\widetilde{l+1}} + r(e_{34})_{\bar{l}} (e_{21})_{\widetilde{l+1}} + r(e_{43})_{\bar{l}} (e_{12})_{\widetilde{l+1}} \right) \\ & - \frac{1}{2} \sum_{l=1}^{n-1} \left(\hat{r}(e_{13})_{\bar{l}} (e_{42})_{\widetilde{l+1}} + \hat{r}(e_{31})_{\bar{l}} (e_{24})_{\widetilde{l+1}} + \hat{r}^{-1}(e_{24})_{\bar{l}} (e_{31})_{\widetilde{l+1}} + \hat{r}^{-1}(e_{42})_{\bar{l}} (e_{13})_{\widetilde{l+1}} \right) \\ & + \frac{1}{4 \cosh i\mu} I_{\tilde{n}} - \frac{\sinh(i\mu)}{4x(0)} (D_{\bar{1}} + C_{\bar{1}}), \end{aligned} \quad (175)$$

where we define

$$\mathbb{I}^{(1)} = e_{11} + e_{44}, \quad \mathbb{I}^{(2)} = e_{22} + e_{33} \quad (176)$$

and they play the role of the ‘unit’ whenever the indices (1,4) and (2,3) are involved respectively. Notice that the first line of the Hamiltonian describes exactly the XXZ model with open boundaries, whereas the second line gives the open XX model. There are also some extra ‘mixing’ terms in the following lines of the form $e_{ij} \otimes e_{\bar{i}\bar{j}}$ ($i \neq j$, $i \neq \bar{j}$). The last two terms come from the right boundary interaction. They are given by

$$D_{\bar{1}} = \sinh i\mu \mathbb{I}^{(1)}, \quad C_{\bar{1}} = \sinh i\mu (S_{\bar{1}}^+ + S_{\bar{1}}^-) - 2 \cosh i\mu S_{\bar{1}}^z. \quad (177)$$

Note finally that the left boundary interaction is trivial $K^+ = \mathbb{I}$, which is why the corresponding term in the Hamiltonian is proportional to unit.

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